SYMMETRY BREAKING FOR TORAL ACTIONS IN SIMPLE MECHANICAL SYSTEMS

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Abstract. For simple mechanical systems, bifurcating branches of relative equilibria with trivial symmetry from a given set of relative equilibria with toral symmetry are found. Lyapunov stability conditions along these branches are given.

Contents

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1. Introduction

This paper investigates the problem of symmetry breaking in the context of simple mechanical systems with compact symmetry Lie group G. Let $\mathbb{T}$ be a maximal torus of G whose Lie algebra is denoted by $\mathfrak{t}$. Denote by Q the configuration space of the mechanical system. Assume that every infinitesimal generator defined by an element of $\mathfrak{t}$ evaluated at a symmetric configuration $q_e \in Q$ whose symmetry subgroup G_{q_e} lies in $\mathbb{T}$ is a relative equilibrium. The goal of this paper is to give sufficient conditions capable to insure the existence of points in this set from which branches of relative equilibria with trivial symmetry will emerge. Sufficient Lyapunov stability conditions along these branches will be given if $G = \mathbb{T}$. The strategy of the method can be roughly described as follows. Denote by $\mathfrak{t}\cdot q_e$ the set of relative equilibria described above. Take a regular element $\mu \in \mathfrak{g}^*$ which happens to be the momentum value of some relative equilibrium in $\mathfrak{t}\cdot q_e$. Choose a one parameter perturbation $\beta(\tau,\mu) \in \mathfrak{g}^*$ of μ that lies in the set of regular points of $\mathfrak{g}^*$, for small values of the parameter $\tau > 0$. Consider the G_{q_e}-representation on the tangent space $T_{q_e}Q$. Let v_{q_e} be an element in the $\{e\}$-stratum of the representation and also in the normal space to the tangent space at q_e to the orbit $G\cdot q_e$. Assume that its norm is small enough in order for v_{q_e} to lie in the open ball centered at the origin $0_{q_e} \in T_{q_e}Q$ where the Riemannian exponential is a diffeomorphism. The curve τv_{q_e} projects by the exponential map to a curve $q_e(\tau)$ in a neighborhood of q_e in Q whose value at $\tau = 0$ is q_e. Note that the isotropy subgroup at every point on this curve, except for $\tau = 0$, is trivial. We shall search for relative equilibria in TQ starting at points of $\mathfrak{t}\cdot q_e$ such that their base curve in Q equals $q_e(\tau)$ and their momentum values are $\beta(\tau,\mu)$. To do this, we shall choose a curve $\xi(\tau, v_{q_e}, \mu) \in \mathfrak{g}$ uniquely determined by $\beta(\tau,\mu)$; as will be explained in the course of the construction, $\xi(\tau, v_{q_e}, \mu)$ equals the value of the inverse of the locked inertial tensor on $\beta(\tau,\mu)$ for $\tau \neq 0$. If one can show that the limit of $\xi(\tau, v_{q_e}, \mu)$ exists and belongs to $\mathfrak{t}$ for $\tau \to 0$, then the infinitesimal generator of this value evaluated at q_e is automatically a relative equilibrium since it belongs to $\mathfrak{t}\cdot q_e$. It will be also shown that the infinitesimal generators of $\xi(\tau, v_{q_e}, \mu)$ evaluated at $q_e(\tau)$ are relative equilibria. This produces a branch of relative equilibria starting at this specific point in $\mathfrak{t}\cdot q_e$ which has trivial isotropy for $\tau > 0$ and which depends smoothly on the additional parameter $\mu \in \mathfrak{g}^*$. In this method, there are two key technical problems, namely, the existence of the limit of $\xi(\tau, v_{q_e}, \mu)$ as $\tau \to 0$ and the extension of the amended potential at points with symmetry. The existence of the limit of $\xi(\tau, v_{q_e}, \mu)$ as $\tau \to 0$ will be shown using the Lyapunov-Schmidt procedure. To extend the amended potential and its derivative at points with symmetry, two auxiliary functions obtained by blow-up will be introduced. The analysis breaks up in two problems on a space orthogonal to the G-orbit. The present paper can be regarded as a sequel to the work of Hernández and Marsden [6]. The main difference is that one single hypothesis from [6] has been retained, namely that all points of $\mathfrak{t}\cdot q_e$ are relative equilibria. We have also eliminated a strong nondegeneracy assumption in [6]. But the general principles of the strategy of the proof having to do with a regularization of the amended potential at points with symmetry, where it is not a priori defined, remains the same. In a future paper we shall further modify this method to deal with bifurcating branches of relative equilibria that have a given isotropy, different from the trivial one, along the branch. The paper is organized as follows. In §2 we quickly review the necessary material on symmetric simple mechanical systems and introduce the notations and conventions for the entire paper. Relative equilibria and their characterizations for general symmetric mechanical systems and for simple ones in terms of the augmented and amended potentials are recalled in §3. Section §4 gives a brief summary of facts from the theory of proper group actions needed in this paper. After these short introductory sections, §5 presents the main bifurcation result of the paper. The existence of branches of relative equilibria starting at certain points in $\mathfrak{t}\cdot q_e$, depending on several parameters and having trivial symmetry off $\mathfrak{t}\cdot q_e$, is proved in Theorem 5.17, the main result of this paper. In §6, using a result

of Patrick [16], Lyapunov stability conditions for these branches are given if the symmetry group is a torus.

2. Lagrangian mechanical systems

This section summarizes the key facts from the theory of Lagrangian systems with symmetry and sets the notations and conventions to be used throughout this paper. The references for this section are [1], [9], [11], [12].

2.1. **Lagrangian mechanical systems with symmetry.** Let Q be a smooth manifold, the configuration space of a mechanical system. The ***fiber derivative*** or ***Legendre transform*** $\mathbb{F}L : TQ \to T^*Q$ of L is a vector bundle map covering the identity defined by

$$\langle \mathbb{F}L(v_q), w_q \rangle = \left.\frac{d}{dt}\right|_{t=0} L(v_q + tw_q)$$

for any $v_q, w_q \in TQ$. The ***energy*** of L is defined by $E(v_q) = \langle \mathbb{F}L(v_q), v_q \rangle - L(v_q)$, $v_q \in T_qQ$. The pull back by $\mathbb{F}L$ of the canonical one– and two–forms of T^*Q give the ***Lagrangian one*** and ***two-forms*** Θ_L and Ω_L on TQ respectively, that have thus the expressions

$$\langle \Theta_L(v_q), \delta v_q \rangle = \langle \mathbb{F}L(v_q), T_{v_q}\pi_Q(\delta v_q) \rangle, \quad v_q \in T_qQ, \quad \delta v_q \in T_{v_q}TQ, \qquad \Omega_L = -\mathbf{d}\Theta_L,$$

where $\pi_Q : TQ \to Q$ is the tangent bundle projection. The Lagrangian L is called ***regular*** if $\mathbb{F}L$ is a local diffeomorphism, which is equivalent to Ω_L being a symplectic form on TQ. The Lagrangian L is called ***hyperregular*** if $\mathbb{F}L$ is a diffeomorphism and hence a vector bundle isomorphism. The ***Lagrangian vector field*** X_E of L is uniquely determined by the equality

$$\Omega_L(v_q)(X_E(v_q), w_q) = \langle \mathbf{d}E(v_q), w_q \rangle, \quad \text{for} \quad v_q,\ w_q \in T_qQ.$$

A ***Lagrangian dynamical system***, or simply a ***Lagrangian system***, for L is the dynamical system defined by X_E, i.e., $\dot{v} = X_E(v)$. In standard coordinates $(q^i, \dot{q}^i)$ the trajectories of X_E are given by the second order equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0,$$

which are the classical the Euler-Lagrange equations. Let $\Psi : G \times Q \to Q$ be a smooth left Lie group action on Q and let $L : TQ \to \mathbb{R}$ be a Lagrangian that is invariant under the lifted action of G to TQ. Denote by $\mathfrak{g}$ the Lie algebra of G. From the definition of the fiber derivative it immediately follows that $\mathbb{F}L$ is equivariant relative to the lifted G–actions to TQ and T^*Q. The G-invariance of L implies that X_E is G-equivariant, that is, $\Psi_g^* X_E = X_E$ for any $g \in G$. The G–action on TQ admits a momentum map given by

$$\langle \mathbf{J}_L(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle, \quad \text{for} \quad v_q \in T_qQ, \quad \xi \in \mathfrak{g}.$$

where $\xi_Q(q) := d\exp(t\xi)\cdot q/dt|_{t=0}$ is the ***infinitesimal generator*** of $\xi \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G. Recall that the momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ on T^*Q is given by

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle, \quad \text{for} \quad \alpha_q \in T_q^*Q, \quad \xi \in \mathfrak{g}$$

and hence $\mathbf{J}_L = \mathbf{J} \circ \mathbb{F}L$. We shall denote by $g \cdot q := \Psi(g, q)$ the action of the element $g \in G$ on the point $q \in Q$. Similarly, the lifted actions of G on TQ and T^*Q are denoted by

$$g \cdot v_q := T_q\Psi_g(v_q) \quad \text{and} \quad g \cdot \alpha_q := T_{g\cdot q}^*\Psi_{g^{-1}}(\alpha_q)$$

for $g \in G$, $v_q \in T_qQ$, and $\alpha_q \in T_q^*Q$.

2.2. **Simple mechanical systems.** A ***simple mechanical system*** $(Q, \langle\langle\cdot,\cdot\rangle\rangle_Q, V)$ consists of a Riemannian manifold $(Q, \langle\langle\cdot,\cdot\rangle\rangle_Q)$ together with a potential function $V : Q \to \mathbb{R}$. These elements define a Hamiltonian system on (T^*Q, ω) with Hamiltonian given by $H : T^*Q \to \mathbb{R}$, $H(\alpha_q) = \frac{1}{2}\langle\langle\alpha_q, \alpha_q\rangle\rangle_{T^*Q} + V(q)$, where $\alpha_q \in T^*_qQ$ and $\langle\langle\cdot,\cdot\rangle\rangle_{T^*Q}$ is the vector bundle metric on T^*Q induced by the Riemannian metric of Q. The Hamiltonian vector field X_H is uniquely given by the relation $\mathbf{i}_{X_H}\omega = \mathbf{d}H$, where ω is the canonical symplectic form on T^*Q. The dynamics of a simple mechanical system can also be described in terms of Lagrangian mechanics, whose description takes place on TQ. The Lagrangian for a simple mechanical system is given by $L : TQ \to \mathbb{R}$, $L(v_q) = \frac{1}{2}\langle\langle v_q, v_q\rangle\rangle_Q - V(q)$, where $v_q \in T_qQ$. The energy of L is $E(v_q) = \frac{1}{2}\langle\langle v_q, v_q\rangle\rangle + V(q)$. Since the fiber derivative for a simple mechanical system is given by $\langle \mathbb{F}L(v_q), w_q\rangle = \langle\langle v_q, w_q\rangle\rangle_Q$, or in local coordinates $\mathbb{F}L\left(\dot q^i \frac{\partial}{\partial q^i}\right) = g_{ij}\dot q^j dq^i$, where g_{ij} is the local expression for the metric on Q, it follows that L is hyperregular. The relationship between the Hamiltonian and the Lagrangian dynamics is the following: the vector bundle isomorphism $\mathbb{F}L$ bijectively maps the trajectories of X_E to the trajectories of X_H, $(\mathbb{F}L)^*X_H = X_E$, and the base integral curves of X_E and X_H coincide.

2.3. **Simple mechanical systems with symmetry.** Let G act on the configuration manifold Q of a simple mechanical system $(Q, \langle\langle\cdot,\cdot\rangle\rangle_Q, V)$ by isometries. The ***locked inertia tensor*** $\mathbb{I} : Q \to \mathcal{L}(\mathfrak{g}, \mathfrak{g}^*)$, where $\mathcal{L}(\mathfrak{g}, \mathfrak{g}^*)$ denotes the vector space of linear maps from $\mathfrak{g}$ to $\mathfrak{g}^*$, is defined by

$$\langle \mathbb{I}(q)\xi, \eta\rangle = \langle\langle \xi_Q(q), \eta_Q(q)\rangle\rangle_Q$$

for any $q \in Q$ and any $\xi, \eta \in \mathfrak{g}$. If the action is ***locally free*** at $q \in Q$, that is, the isotropy subgroup G_q is discrete, then $\mathbb{I}(q)$ is an isomorphism and hence defines an inner product on $\mathfrak{g}$. In general, the defining formula of $\mathbb{I}(q)$ shows that $\ker \mathbb{I}(q) = \mathfrak{g}_q := \{\xi \in \mathfrak{g} \mid \xi_Q(q) = 0\}$. Suppose the action is locally free at every point $q \in Q$. Then on can define the ***mechanical connection*** $\mathcal{A} \in \Omega^1(Q; \mathfrak{g})$ by

$$\mathcal{A}(q)(v_q) = \mathbb{I}(q)^{-1}\mathbf{J}_L(v_q), \quad v_q \in T_qQ.$$

If the G–action is free and proper, so $Q \to Q/G$ is a G–principal bundle, then $\mathcal{A}$ is a connection one–form on the principal bundle $Q \to Q/G$, that is, it satisfies the following properties:

- $\mathcal{A}(q) : T_qQ \to \mathfrak{g}$ is linear and G-equivariant for every $q \in Q$, which means that
$$\mathcal{A}(g\cdot q)(g\cdot v_q) = \mathrm{Ad}_g[\mathcal{A}(q)(v_q)],$$
for any $v_q \in T_qQ$ and any $g \in G$, where Ad denotes the adjoint representation of G on $\mathfrak{g}$;
- $\mathcal{A}(q)(\xi_Q(q)) = \xi$, for any $\xi \in \mathfrak{g}$.

If $\mu \in \mathfrak{g}^*$ is given, we denote by $\mathcal{A}_\mu \in \Omega^1(Q)$ the μ–component of $\mathcal{A}$, that is, the one–form on Q defined by $\langle \mathcal{A}_\mu(q), v_q\rangle = \langle \mu, \mathcal{A}(q)(v_q)\rangle$ for any $v_q \in T_qQ$. The G-invariance of the metric and the relation

$$(\mathrm{Ad}_g\,\xi)_Q(q) = g\cdot \xi_Q(g^{-1}\cdot q),$$

implies that

$$\mathbb{I}(g\cdot q) = \mathrm{Ad}^*_{g^{-1}} \circ \mathbb{I}(q) \circ \mathrm{Ad}_{g^{-1}}\,. \tag{2.1}$$

We shall also need later the infinitesimal version of the above identity

$$T_q\mathbb{I}\left(\xi_Q(q)\right) = -\,\mathrm{ad}^*_\xi \circ \mathbb{I}(q) - \mathbb{I}(q)\circ \mathrm{ad}_\xi, \tag{2.2}$$

which implies

$$\langle T_q\mathbb{I}(\zeta_Q(q))\xi, \eta\rangle = \mathbf{d}\langle \mathbb{I}(\cdot)\xi, \eta\rangle(q)\left(\zeta_Q(q)\right) = \langle \mathbb{I}(q)[\xi,\zeta], \eta\rangle + \langle \mathbb{I}(q)\xi, [\eta,\zeta]\rangle. \tag{2.3}$$

for all $q \in Q$ and all $\xi, \eta, \zeta \in \mathfrak{g}$.

3. Relative equilibria

This section recalls the basic facts about relative equilibria that will be needed in this paper. For proofs see [1], [9], [11], [12], [19].

3.1. **Basic definitions and concepts.** Let $\Psi : G \times Q \to Q$ be a left action of the Lie group on the manifold Q. A vector field $X : Q \to TQ$ is said to be ***G-equivariant*** if

$$T_q\Psi_g(X(q)) = X(\Psi_g(q)) \quad \text{or, equivalently,} \quad \Psi_g^* X = X$$

for all $q \in Q$ and $g \in G$. If X is G-equivariant, then G is said to be a ***symmetry group*** of the dynamical system $\dot{q} = X(q)$. A ***relative equilibrium*** of a G–equivariant vector field X is a point $q_e \in Q$ at which the value of X coincides with the infinitesimal generator of some element $\xi \in \mathfrak{g}$, usually called the ***velocity*** of q_e, i.e.,

$$X(q_e) = \xi_Q(q_e).$$

A relative equilibrium q_e is said to be ***asymmetric*** if the isotropy subalgebra $\mathfrak{g}_{q_e} := \{\eta \in \mathfrak{g} \mid \eta_Q(q_e) = 0\} = \{0\}$, and ***symmetric*** otherwise. Note that if q_e is a relative equilibrium with velocity $\xi \in \mathfrak{g}$, then for any $g \in G$, $g \cdot q_e$ is a relative equilibrium with velocity $\mathrm{Ad}_g \xi$. The flow of an equivariant vector field induces a flow on the quotient space. Thus, if the G–action is free and proper, a relative equilibrium defines an equilibrium of the induced vector field on the quotient space and conversely, any element in the fiber over an equilibrium in the quotient space is a relative equilibrium of the original system.

3.2. **Relative equilibria in Hamiltonian G-systems.** Given is a symplectic manifold (P, ω), a left Lie group action of G on P that admits a momentum map $\mathbf{J} : P \to \mathfrak{g}^*$, that is, $X_{\mathbf{J}^\xi} = \xi_P$, for any $\xi \in \mathfrak{g}$, where $\mathbf{J}^\xi(p) := \langle \mathbf{J}(p), \xi \rangle$, $p \in P$, is the ξ–component of $\mathbf{J}$. We shall also assume throughout this paper that the momentum map $\mathbf{J}$ is equivariant, that is, $\mathbf{J}(g \cdot p) = \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(p)$, for any $g \in G$ and any $p \in P$. Given is also a G–invariant function $H : P \to \mathbb{R}$. Noether's theorem states that the $\mathbf{J}$ is conserved along the flow F_t of the Hamiltonian vector field X_H. In what follows we shall call the quadruple $(Q, \omega, H, \mathbf{J}, G)$ a ***Hamiltonian G–system***. Consistent with the general definition presented above, a point $p_e \in P$ is a ***relative equilibrium*** if

$$X_H(p_e) \in T_{p_e}(G \cdot p_e),$$

where $G \cdot p_e : \{g \cdot p_e \mid g \in G\}$ denotes the G–orbit through p_e. Relative equilibria are characterized in the following manner.

Proposition 3.1. *(**Characterization of relative equilibria**). Let $p_e \in P$ and $p_e(t)$ be the integral curve of X_H with initial condition $p_e(0) = p_e$. Let $\mu := \mathbf{J}(p_e)$. Then the following are equivalent:*

- **(i)** *p_e is a relative equilibrium.*
- **(ii)** *There exists $\xi \in \mathfrak{g}$ such that $p_e(t) = \exp(t\xi) \cdot p_e$.*
- **(iii)** *There exists $\xi \in \mathfrak{g}$ such that p_e is a critical point of the **augmented Hamiltonian***

$$H_\xi(p) := H(p) - \langle \mathbf{J}(p) - \mu, \xi \rangle.$$

Once we have a relative equilibrium, its entire G-orbit consists of relative equilibria and the relation between the velocities of the relative equilibria that are on the same G-orbit is given by the adjoint action of G on $\mathfrak{g}$.

Proposition 3.2. *With the notations of the previous proposition, let p_e be a relative equilibrium with velocity ξ. Then*

- **(i)** *for any $g \in G$, $g \cdot q_e$ is also a relative equilibrium whose velocity is $\mathrm{Ad}_g \xi$;*
- **(ii)** *$\xi(q_e) \in \mathfrak{g}_\mu := \{\eta \in \mathfrak{g} \mid \mathrm{ad}^*_\eta \mu = 0\}$, the coadjoint isotropy subalgebra at $\mu \in \mathfrak{g}^*$, i.e., $\mathrm{Ad}^*_{\exp t\xi} \mu = \mu$ for any $t \in \mathbb{R}$.*

3.3. **Relative equilibria in simple mechanical G-systems.** In the case of simple mechanical G-systems, the characterization (**iii**) in Proposition 3.1 can be simplified in such way that the search of relative equilibria reduces to the search of critical points of a real valued function on Q. Depending on whether one keeps track of the velocity or the momentum of a relative equilibrium, this simplification yields the *augmented* or the *amended* potential criterion, which we introduce in what follows. Let $(Q, \langle\!\langle \cdot, \cdot \rangle\!\rangle_Q, V, G)$ be a simple mechanical G–system.

- For $\xi \in \mathfrak{g}$, the ***augmented potential*** $V_\xi : Q \to \mathbb{R}$ is defined by $V_\xi(q) := V(q) - \frac{1}{2}\langle \mathbb{I}(q)\xi, \xi \rangle$.
- For $\mu \in \mathfrak{g}^*$, the ***amened potential*** $V_\mu : Q \to \mathbb{R}$ is defined by $V_\mu(q) := V(q) + \frac{1}{2}\langle \mu, \mathbb{I}(q)^{-1}\mu \rangle$.

Note that the amended potential is defined at $q \in Q$ only if q in an asymmetric point. There is an alternate expression for the amended potential, namely, $V_\mu(q) = (H \circ \mathcal{A}_\mu)(q)$.

Proposition 3.3. *(**Augmented potential criterion**). A point $(q_e, p_e) \in T^*Q$ is a relative equilibrium if and only if there exists a $\xi \in \mathfrak{g}$ such that:*

(i) $p_e = \mathbb{F}L(\xi_Q(q_e))$ *and*
(ii) q_e *is a critical point of* V_ξ.

Proposition 3.4. *(**Amended potential criterion**). A point $(q_e, p_e) \in T^*Q$ is a relative equilibrium if and only if there exists a $\mu \in \mathfrak{g}^*$ such that:*

(i) $p_e = \mathcal{A}_\mu(q_e)$ *and*
(ii) q_e *is a critical point of* V_μ.

4. Some basic results from the theory of Lie group actions

We shall need a few fundamental results form the theory of group actions which we now review. For proofs and further information see [3], [4], [7], [15].

4.1. **Maximal tori.** Let V be a representation space of a compact Lie group G. A point $v \in V$ is ***regular*** if there is no G–orbit in V whose dimension is strictly greater than the dimension of the G–orbit through v. The set of regular points, denoted V_{reg}, is open and dense in V. In particular, $\mathfrak{g}_{reg}$ and $\mathfrak{g}^*_{reg}$, denote the set of regular points in $\mathfrak{g}$ and $\mathfrak{g}^*$ with respect to adjoint and coadjoint representation, respectively. A subgroup of a Lie group is said to be a ***torus*** if it is isomorphic to $S^1 \times \cdots \times S^1$. Every Abelian subgroup of a compact connected Lie group is isomorphic to a torus. A subgroup of a Lie group is said to be a ***maximal torus*** if it is a torus that is not properly contained in some other torus. Every $\xi \in \mathfrak{g}$ belongs to at least one maximal Abelian subalgebra and every $\xi \in \mathfrak{g} \cap \mathfrak{g}_{reg}$ belongs to exactly one such maximal Abelian subalgebra. Every maximal Abelian subalgebra is the Lie algebra of some maximal torus in G. Let $\mathfrak{t}$ be the maximal Abelian subalgebra corresponding to a maximal torus T. Then for any $\xi \in \mathfrak{t} \cap \mathfrak{g}_{reg}$, we have that $G_\xi = T$. The space $[\mathfrak{g}, \mathfrak{t}]$ is the orthogonal complement to $\mathfrak{t}$ in $\mathfrak{g}$ with respect to any G–invariant inner product on $\mathfrak{g}$. Such an inner product exists by compactness of G by simply averaging any inner product on $\mathfrak{g}$. Therefore, we have $\mathfrak{g} = \mathfrak{t} \oplus [\mathfrak{g}, \mathfrak{t}]$. Let $[\mathfrak{g}, \mathfrak{t}]^\circ$ the annihilator of $[\mathfrak{g}, \mathfrak{t}]$. Then $G_\mu = T$ for every $\mu \in [\mathfrak{g}, \mathfrak{t}]^\circ \cap \mathfrak{g}^*_{reg}$. Since $[\mathfrak{g}, \mathfrak{t}]^\circ \cap \mathfrak{g}^*_{reg}$ is dense in $[\mathfrak{g}, \mathfrak{t}]^\circ$, it follows that $T \subset G_\mu$ for every $\mu \in [\mathfrak{g}, \mathfrak{t}]^\circ$.

4.2. **Twisted products.** Let G be a Lie group and $H \subset G$ be a Lie subgroup. Suppose that H acts on the left on a manifold A. The ***twisted action*** of H on the product $G \times A$ is defined by

$$h \cdot (g, a) = (gh, h^{-1} \cdot a), \quad h \in H, \quad g \in G, \quad a \in A.$$

Note that this action is free and proper by the freeness and properness of the action on the G–factor. The ***twisted product*** $G \times_H A$ is defined as the orbit space $(G \times A)/H$ of the twisted action. The elements of $G \times_H A$ will be denoted by $[g, a]$, $g \in G$, $a \in A$. The twisted product $G \times_H A$ is a G–space

relative to the left action defined by $g' \cdot [g, a] = [g'g, a]$. Also, the action of H on A is proper if and only if the G–action on $G \times_H A$ is proper. The isotropy subgroups of the G–action on the twisted product $G \times_H A$ satisfy

$$G_{[g,a]} = gH_a g^{-1}, \quad g \in G, \quad a \in A.$$

4.3. **Slices.** Throughout this paragraph it will be assumed that $\Psi : G \times Q \to Q$ is a left proper action of the Lie group G on the manifold Q. This action will not be assumed to be free, in general. For $q \in Q$ we will denote by $H := G_q := \{g \in G \mid g \cdot q = q\}$ the isotropy subgroup of the action Ψ at q. We shall introduce also the following convenient notation: if $K \subset G$ is a Lie subgroup of G (possibly equal to G), $\mathfrak{k}$ is its Lie algebra, and $q \in Q$, then $\mathfrak{k} \cdot q := \{\eta_Q(q) \mid \eta \in \mathfrak{k}\}$ is the tangent space to the orbit $K \cdot q$ at q. A ***tube*** around the orbit $G \cdot q$ is a G-equivariant diffeomorphism $\varphi : G \times_H A \to U$, where U is a G-invariant neighborhood of $G \cdot q$ and A is some manifold on which H acts. Note that the G-action on the twisted product $G \times_H A$ is proper since the isotropy subgroup H is compact and, consequently, its action on A is proper. Hence the G-action on $G \times_H A$ is proper. Let S be a submanifold of Q such that $q \in S$ and $H \cdot S = S$. We say that S is a ***slice*** at q if the map

$$\varphi : G \times_H S \to U$$

$$[g, s] \mapsto g \cdot s$$

is a tube about $G \cdot q$, for some G–invariant open neighborhood of $G \cdot q$. Notice that if S is a slice at q then $g \cdot S$ is a slice at the point $g \cdot q$. The following statements are equivalent:

(i) There is a tube $\varphi : G \times_H A \to U$ about $G \cdot q$ such that $\varphi([e, A]) = S$.
(ii) S is a slice at q.
(iii) The submanifold S satisfies the following properties:
  (a) The set $G \cdot S$ is an open neighborhood of the orbit $G \cdot q$ and S is closed in $G \cdot S$.
  (b) For any $s \in S$ we have $T_sQ = \mathfrak{g} \cdot s + T_sS$. Moreover, $\mathfrak{g} \cdot s \cap T_sS = \mathfrak{h} \cdot s$, where $\mathfrak{h}$ is the Lie algebra of H. In particular $T_qQ = \mathfrak{g} \cdot q \oplus T_qS$.
  (c) S is H-invariant. Moreover, if $s \in S$ and $g \in G$ are such that $g \cdot s \in S$, then $g \in H$.
  (d) Let $\sigma : U \subset G/H \to G$ be a local section of the submersion $G \to G/H$. Then the map $F : U \times S \to Q$ given by $F(u, s) := \sigma(u) \cdot s$ is a diffeomorphism onto an open set of Q.
(iv) $G \cdot S$ is an open neighborhood of $G \cdot q$ and there is an equivariant smooth retraction

$$r : G \cdot S \to G \cdot q$$

of the injection $G \cdot q \hookrightarrow G \cdot S$ such that $r^{-1}(q) = S$.

Theorem 4.1. ***(Slice Theorem)*** *Let Q be a manifold and G be a Lie group acting properly on Q at the point $q \in Q$. Then, there exists a slice for the G–action at q.*

Theorem 4.2. ***(Tube Theorem)*** *Let Q be a manifold and G be a Lie group acting properly on Q at the point $q \in Q$, $H := G_q$. There exists a tube $\varphi : G \times_H B \to U$ about $G \cdot q$ such that $\varphi([e, 0]) = q$, $\varphi([e, B]) =:$ S is a slice at q; B is an open H–invariant neighborhood of 0 in the vector space $T_qQ/T_q(G \cdot q)$, on which H acts linearly by $h \cdot (v_q + T_q(G \cdot q)) := T_q\Psi_h(v_q) + T_q(G \cdot q)$.*

If Q is a Riemannian manifold then B can be chosen to be a G_q–invariant neighborhood of 0 in $(\mathfrak{g} \cdot q)^\perp$, the orthogonal complement to $\mathfrak{g} \cdot q$ in T_qQ. In this case $U = G \cdot \mathrm{Exp}_q(B)$, where $\mathrm{Exp}_q : T_qQ \to Q$ is the Riemannian exponential map.

4.4. **Type submanifolds and fixed point subspaces.** Let G be a Lie group acting on a manifold Q. Let H be a closed subgroup of G. We define the following subsets of Q :

$$\begin{aligned} Q_{(H)} &= \{q \in Q \mid G_q = gHg^{-1}, g \in G\}, \\ Q^H &= \{q \in Q \mid H \subset G_q\}, \\ Q_H &= \{q \in Q \mid H = G_q\}. \end{aligned}$$

All these sets are submanifolds of Q. The set $Q_{(H)}$ is called the (H)–***orbit type submanifold***, Q_H is the H–***isotropy type submanifold***, and Q^H is the H–***fixed point submanifold***. We will collectively call these subsets the ***type submanifolds***. We have:

- Q^H is closed in Q;
- $Q_{(H)} = G \cdot Q_H$;
- Q_H is open in Q^H.
- the tangent space at $q \in Q_H$ to Q_H equals
$$T_qQ_H = \{v_q \in T_qQ \mid T_q\Psi_h(v_q) = v_q, \forall h \in H\} = (T_qQ)^H = T_qQ^H;$$
- $T_q(G \cdot q) \cap (T_qQ)^H = T_q(N(H) \cdot q)$, where $N(H)$ is the normalizer of H in G;
- if H is compact then $Q_H = Q^H \cap Q_{(H)}$ and Q_H is closed in $Q_{(H)}$.

If Q is a vector space on which H acts linearly, the set Q^H is found in the physics literature under the names of ***space of singlets*** or ***space of invariant vectors***.

Theorem 4.3. ***(The stratification theorem)***. *Let Q be a smooth manifold and G be a Lie group acting properly on it. The connected components of the orbit type manifolds $Q_{(H)}$ and their projections onto orbit space $Q_{(H)}/G$ constitute a Whitney stratification of Q and Q/G, respectively. This stratification of Q/G is minimal among all Whitney stratifications of Q/G.*

The proof of this result, that can be found in [4] or [17], is based on the Slice Theorem and on a series of extremely important properties of the orbit type manifolds decomposition that we enumerate in what follows. We start by recalling that the set of conjugacy classes of subgroups of a Lie group G admits a partial order by defining $(K) \preceq (H)$ if and only if H is conjugate to a subgroup of K. Also, a point $q \in Q$ in a proper G–space Q (or its corresponding G–orbit, $G \cdot q$) is called ***principal*** if its corresponding local orbit type manifold is open in Q. The orbit $G \cdot q$ is called ***regular*** if the dimension of the orbits nearby coincides with the dimension of $G \cdot q$. The set of principal and regular orbits will be denoted by Q_{princ}/G and Q_{reg}/G, respectively. Using this notation we have:

- For any $q \in Q$ there exists an neighborhood U of q that intersects only finitely many connected components of finitely many orbit type manifolds. If Q is compact or a linear space where G acts linearly, then the G–action on Q has only finitely many distinct connected components of orbit type manifolds.
- For any $q \in Q$ there exists an open neighborhood U of q such that $(G_q) \preceq (G_x)$, for all $x \in U$. In particular, this implies that $\dim G \cdot q \leq \dim G \cdot x$, for all $x \in U$.
- ***Principal Orbit Theorem***: For every connected component Q^0 of Q the subset $Q_{reg} \cap Q^0$ is connected, open, and dense in Q^0. Each connected component $(Q/G)^0$ of Q/G contains only one principal orbit type, which is connected open and dense in $(Q/G)^0$.

5. Regularization of the amended potential criterion

In this section we shall follow the strategy in [6] to give sufficient criteria for finding relative equilibria emanating from a given one and to find a method that distinguishes between the distinct branches. The criterion will involve a certain regularization of the amended potential. The main difference with [6]

is that all hypotheses but one have been eliminated and we work with a general torus and not just a circle. The conventions, notations, and method of proof are those in [6].

5.1. **The bifurcation problem.** Let $(Q, \langle\!\langle\cdot,\cdot\rangle\!\rangle_Q, V, G)$ be a simple mechanical G-system, with G a compact Lie group with the Lie algebra $\mathfrak{g}$. Recall that the left G–action $\Psi : G \times Q \to Q$ is by isometries and that the potential $V : Q \to \mathbb{R}$ is G–invariant. Let $q_e \in Q$ be a symmetric point whose isotropy group $G_{q_e} \subset \mathbb{T}$ is contained in a maximal torus $\mathbb{T}$ of G. Denote by $\mathfrak{t} \subset \mathfrak{g}$ the Lie algebra of $\mathbb{T}$; thus $\mathfrak{t}$ is a maximal Abelian Lie subalgebra of $\mathfrak{g}$. Throughout this section we shall make the following hypothesis:

(H) *every $v_{q_e} \in \mathfrak{t} \cdot q_e$ is a relative equilibrium.*

The following result was communicated to us by J. Montaldi.

Proposition 5.1. *In the context above we have that:*

$$\begin{array}{ll} (i) & \mathbf{d}V(q_e) = 0 \\ (ii) & \mathbb{I}(q_e)\mathfrak{t} \subseteq [\mathfrak{g}, \mathfrak{t}]^\circ. \end{array}$$

Proof. (i) Because all the elements in $\mathfrak{t} \cdot q_e$ are relative equilibria, we have by the augmented potential criterion $\mathbf{d}V_\xi(q_e) = 0$, for any $\xi \in \mathfrak{t}$. Consequently for $\xi = 0$ we will obtain $0 = \mathbf{d}V_0(q_e) = \mathbf{d}V(q_e)$. (ii) Substituting in the relation (2.3), q by q_e and setting $\eta = \xi \in \mathfrak{t}$ we obtain:

$$\mathbf{d}\langle \mathbb{I}(\,\cdot\,)\xi, \xi\rangle(q_e)(\zeta_Q(q_e)) = \langle \mathbb{I}(q_e)[\xi,\zeta], \xi\rangle + \langle \mathbb{I}(q_e)\xi, [\xi,\zeta]\rangle = 2\langle \mathbb{I}(q_e)\xi, [\xi,\zeta]\rangle$$

for any $\xi \in \mathfrak{t}$ and $\zeta \in \mathfrak{g}$. The augmented potential criterion yields

$$0 = \mathbf{d}V_\xi(q_e) = \mathbf{d}V(q_e) - \frac{1}{2}\mathbf{d}\langle \mathbb{I}(\,\cdot\,)\xi, \xi\rangle(q_e).$$

Since $\mathbf{d}V(q_e) = 0$ by (i), this implies $\mathbf{d}\langle \mathbb{I}(\,\cdot\,)\xi, \xi\rangle(q_e) = 0$ and consequently $\langle \mathbb{I}(q_e)\xi, [\xi,\zeta]\rangle = 0$, for any $\xi \in \mathfrak{t}$ and $\zeta \in \mathfrak{g}$. So we have the inclusion

$$\mathbb{I}(q_e)\xi \subseteq [\mathfrak{g}, \xi]^\circ.$$

Now we will prove that $[\mathfrak{g},\xi]^\circ = [\mathfrak{g},\mathfrak{t}]^\circ$ for regular elements $\xi \in \mathfrak{t}$. For this it is enough to prove that $[\xi,\mathfrak{g}] = [\mathfrak{t},\mathfrak{g}]$ for regular elements $\xi \in \mathfrak{t}$. It is obvious that $[\xi,\mathfrak{g}] \subseteq [\mathfrak{t},\mathfrak{g}]$. Equality will follow by showing that both spaces have the same dimension. To do this, let $F_\xi : \mathfrak{g} \to \mathfrak{g}$, $F_\xi(\eta) := \mathrm{ad}_\xi\, \eta$, which is obviously a linear map whose image and kernel are $\mathrm{Im}(F_\xi) = [\xi,\mathfrak{g}]$ and $\ker(F_\xi) = \mathfrak{g}_\xi$. Because $\xi \in \mathfrak{t}$ is a regular element we have that $\mathfrak{g}_\xi = \mathfrak{t}$ and so $\ker(F_\xi) = \mathfrak{t}$. Thus $\dim(\mathfrak{g}) = \dim(\mathfrak{t}) + \dim([\xi,\mathfrak{g}])$ and so using the fact that $\dim(\mathfrak{g}) = \dim(\mathfrak{t}) + \dim([\mathfrak{t},\mathfrak{g}])$ (since $\mathfrak{g} = \mathfrak{t} \oplus [\mathfrak{g},\mathfrak{t}]$, $\mathfrak{g}$ being a compact Lie algebra), we obtain the equality $\dim([\xi,\mathfrak{g}]) = \dim([\mathfrak{t},\mathfrak{g}])$. Therefore, $[\xi,\mathfrak{g}] = [\mathfrak{t},\mathfrak{g}]$ for any regular element $\xi \in \mathfrak{t}$. Summarizing, we proved

$$\mathbb{I}(q_e)\xi \subseteq [\mathfrak{g}, \mathfrak{t}]^\circ,$$

for any regular element $\xi \in \mathfrak{t}$. The continuity of $\mathbb{I}(q_e)$, the closedness of $[\mathfrak{g},\mathfrak{t}]^\circ$, and that fact that the regular elements $\xi \in \mathfrak{t}$ form a dense subset of $\mathfrak{t}$, implies that

$$\mathbb{I}(q_e)\xi \subseteq [\mathfrak{g}, \mathfrak{t}]^\circ,$$

for any $\xi \in \mathfrak{t}$ and hence $\mathbb{I}(q_e)\mathfrak{t} \subseteq [\mathfrak{g},\mathfrak{t}]^\circ$. ∎

Lemma 5.2. *For each $v_{q_e} \in \mathfrak{t} \cdot q_e$ we have $G_{v_{q_e}} = G_{q_e}$.*

Proof. The inclusion $G_{v_{q_e}} \subseteq G_{q_e}$ is obviously true, so it will be enough to prove that $G_{v_{q_e}} \supseteq G_{q_e}$. To see this, let $g \in G_{q_e}$ and $v_{q_e} = \xi_Q(q_e) \in \mathfrak{t} \cdot q_e$, with $\xi \in \mathfrak{t}$. Then, since G_{q_e} is Abelian, we get

$$\begin{aligned} T_{q_e}\Psi_g\left(v_{q_e}\right) &= T_{q_e}\Psi_g\left(\xi_Q(q_e)\right) = T_{q_e}\Psi_g\left(\left.\frac{d}{dt}\right|_{t=0}\Psi_{\exp(t\xi)}(q_e)\right) \\ &= \left.\frac{d}{dt}\right|_{t=0}\left(\Psi_g\circ\Psi_{\exp(t\xi)}\right)(q_e) = \left.\frac{d}{dt}\right|_{t=0}(\Psi_{\exp(t\xi)}\circ\Psi_g)(q_e) \\ &= \left.\frac{d}{dt}\right|_{t=0}\Psi_{\exp(t\xi)}(q_e) = \xi_Q(q_e) = v_{q_e}, \end{aligned}$$

that is, $g \cdot v_{q_e} = v_{q_e}$, as required. □

The bifurcation problem for relative equilibria on TQ can be regarded as a bifurcation problem on the space $Q \times \mathfrak{g}^*$ as the following shows.

Proposition 5.3. *The map $f : TQ \to Q \times \mathfrak{g}^*$ given by $v_q \mapsto (q, \mathbf{J}_L(v_q))$ restricted to the set of relative equilibria is one to one and onto its image.*

Proof. The only thing to be proved is that the map is injective. To see this, let $(q_1, (\xi_1)_Q(q_1))$ and $(q_2, (\xi_2)_Q(q_2))$ be two relative equilibria such that $f(q_1, (\xi_1)_Q(q_1)) = f(q_2, (\xi_2)_Q(q_2))$. Then $q_1 = q_2 =: q$ and $\mathbf{J}_L(q, (\xi_1 - \xi_2)_Q(q)) = \mathbb{I}(q)(\xi_1 - \xi_2) = 0$ which shows that $\xi_1 - \xi_2 \in \ker \mathbb{I}(q) = \mathfrak{g}_q$ and hence $(\xi_1)_Q(q) = (\xi_2)_Q(q)$. □

We can thus change the problem: instead of searching for relative equilibria of the simple mechanical system in TQ, we shall set up a bifurcation problem on $Q \times \mathfrak{g}^*$ such that the image of the relative equilibria by the map f is precisely the bifurcating set. To do this, we begin with some geometric considerations. We construct a G-invariant tubular neighborhood of the orbit $G \cdot q_e$ such that the isotropy group of every point in this neighborhood is a subgroup of G_{q_e}. This follows from the Tube Theorem 4.2. Indeed, let $B \subset (\mathfrak{g} \cdot q_e)^\perp$ be a G_{q_e}-invariant open neighborhood of $0_{q_e} \in (\mathfrak{g} \cdot q_e)^\perp$ such that on the open G-invariant neighborhood $G \cdot \mathrm{Exp}_{q_e}(B)$ of $G \cdot q_e$, we have $(G_{q_e}) \preceq (G_q)$ for every $q \in G \cdot \mathrm{Exp}_{q_e}(B)$. Moreover G acts freely on $G \cdot \mathrm{Exp}_{q_e}\left(B \cap (T_{q_e}Q)_{\{e\}}\right)$. It is easy to see that $B \times \mathfrak{g}^*$ can be identified with a slice at $(q_e, 0)$ with respect to the diagonal action of G on $(G \cdot \mathrm{Exp}_{q_e}(B)) \times \mathfrak{g}^*$. The strategy to prove the existence of a bifurcating branch of relative equilibria with no symmetry from the set of relative equilibria $\mathfrak{t} \cdot q_e$ is the following. Note that we do not know a priori which relative equilibrium in $\mathfrak{t} \cdot q_e$ will bifurcate. We search for a local bifurcating branch of relative equilibria in the following manner. Take a vector $v_{q_e} \in B \cap (T_{q_e}Q)_{\{e\}}$ and note that $\mathrm{Exp}_{q_e}(v_{q_e}) \in Q$ is a point with no symmetry, that is, $G_{\mathrm{Exp}_{q_e}(v_{q_e})} = \{e\}$. Then $\tau v_{q_e} \in B \cap (T_{q_e}Q)_{\{e\}}$, for $\tau \in I$, where I is an open interval containing $[0, 1]$, and $\mathrm{Exp}_{q_e}(\tau v_{q_e})$ is a smooth path connecting q_e, the base point of the relative equilibrium in $\mathfrak{t} \cdot q_e$ containing the branch of bifurcating relative equilibria, to $\mathrm{Exp}_{q_e}(v_{q_e}) \in Q$. In addition, we shall impose that the entire path $\mathrm{Exp}_{q_e}(\tau v_{q_e})$ be formed by base points of relative equilibria. We still need the vector part of these relative equilibria which we postulate to be of the form $\zeta(\tau)_Q(\mathrm{Exp}_{q_e}(\tau v_{q_e}))$, where $\zeta(\tau) \in \mathfrak{g}$ is a smooth path of Lie algebra elements with $\zeta(0) \in \mathfrak{t}$. Since $\mathrm{Exp}_{q_e}(\tau v_{q_e})$ has no symmetry for $\tau > 0$, the locked inertia tensor is invertible at these points and the path $\zeta(\tau)$ will be of the form

$$\zeta(\tau) = \mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))^{-1}(\beta(\tau)),$$

where $\beta(\tau)$ is a smooth path in $\mathfrak{g}^*$ with $\beta(0) \in \mathbb{I}(q_e)\mathfrak{t}$. Now we shall use the characterization of relative equilibria involving the amended potential to require that the path $\left(\mathrm{Exp}_{q_e}(\tau v_{q_e}), \beta(\tau)\right) \in (G \cdot \mathrm{Exp}_{q_e}(B)) \times \mathfrak{g}^*$ be such that $f^{-1}\left(\left(\mathrm{Exp}_{q_e}(\tau v_{q_e}), \beta(\tau)\right)\right)$ are all relative equilibria. The amended potential

criterion is applicable along the path $\operatorname{Exp}_{q_e}(\tau v_{q_e})$ for $\tau > 0$, because these points have no symmetry. As we shall see below, we shall look for $\beta(\tau)$ of a certain form and then the characterization of relative equilibria via the amended potential will impose conditions on both $\beta(\tau)$ and v_{q_e}. We begin by specifying the form of $\beta(\tau)$.

5.2. **Splittings.** We shall need below certain direct sum decompositions of $\mathfrak{g}$ and $\mathfrak{g}^*$. The compactness of G implies that $\mathfrak{g}$ has an invariant inner product and that $\mathfrak{g} = \mathfrak{t} \oplus [\mathfrak{g}, \mathfrak{t}]$ is an orthogonal direct sum. Let $\mathfrak{k}_1 \subset \mathfrak{t}$ be the orthogonal complement to $\mathfrak{k}_0 := \mathfrak{g}_{q_e}$ in $\mathfrak{t}$. Denoting $\mathfrak{k}_2 := [\mathfrak{g}, \mathfrak{t}]$ we obtain the orthogonal direct sum $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2$. For the dual of the Lie algebra, let $\mathfrak{m}_i := (\mathfrak{k}_j \oplus \mathfrak{k}_k)^\circ$ where (i, j, k) is a cyclic permutation of $(0, 1, 2)$. Then $\mathfrak{g}^* = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is also an orthogonal direct sum relative to the inner product on $\mathfrak{g}^*$ naturally induced by the invariant inner product on $\mathfrak{g}$.

Lemma 5.4. *The subspaces defined by the above splittings have the following properties:*

- **(i)** $\mathfrak{k}_0$, $\mathfrak{k}_1$, $\mathfrak{k}_2$ *are* G_{q_e}*-invariant and* G_{q_e} *acts trivially on* $\mathfrak{k}_0$ *and* $\mathfrak{k}_1$*;*
- **(ii)** $\mathfrak{m}_0$, $\mathfrak{m}_1$, $\mathfrak{m}_2$ *are* G_{q_e}*-invariant and* G_{q_e} *acts trivially on* $\mathfrak{m}_0$ *and* $\mathfrak{m}_1$*.*

Proof. **(i)** Because G_{q_e} is a subgroup of $\mathbb{T}$ it is obvious that G_{q_e} acts trivially on $\mathfrak{t} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ and hence on each summand. To prove the G_{q_e}-invariance of $\mathfrak{k}_2 = [\mathfrak{g}, \mathfrak{t}]$, we use the fact that $\operatorname{Ad}_g[\xi_1, \xi_2] = [\operatorname{Ad}_g \xi_1, \operatorname{Ad}_g \xi_2]$, for any $\xi_1, \xi_2 \in \mathfrak{g}$ and $g \in G$. Indeed, if $\xi_1 \in \mathfrak{g}$, $\xi_2 \in \mathfrak{t}$, $g \in G_{q_e}$ we get $\operatorname{Ad}_g[\xi_1, \xi_2] \in [\mathfrak{g}, \mathfrak{t}] = \mathfrak{k}_2$. **(ii)** For $g \in G_{q_e}$, $\mu \in \mathfrak{m}_0$ we have to prove that $\operatorname{Ad}_g^* \mu \in \mathfrak{m}_0$. Indeed, if $\xi = \xi_1 + \xi_2 \in \mathfrak{k}_1 \oplus \mathfrak{k}_2$, we have

$$\begin{aligned} \langle \operatorname{Ad}_g^* \mu, \xi \rangle &= \langle \operatorname{Ad}_g^* \mu, \xi_1 + \xi_2 \rangle = \langle \mu, \operatorname{Ad}_g(\xi_1 + \xi_2) \rangle \\ &= \langle \mu, \xi_1 + \operatorname{Ad}_g \xi_2 \rangle = 0 \end{aligned}$$

since G_{q_e} acts trivially on $\mathfrak{k}_1$, $\mathfrak{k}_2$ is G_{q_e}–invariant and $\mathfrak{m}_0 = (\mathfrak{k}_1 \oplus \mathfrak{k}_2)^\circ$. The same type of proof holds for $\mathfrak{m}_1$and $\mathfrak{m}_2$. For $g \in G_{q_e}$, $\mu \in \mathfrak{m}_0$ we have to prove that $\operatorname{Ad}_g^* \mu = \mu$. Let $\xi = \xi_0 + \xi_1 + \xi_2 \in \mathfrak{g}$, with $\xi_i \in \mathfrak{k}_i$, $i = 0, 1, 2$. We have

$$\begin{aligned} \langle \operatorname{Ad}_g^* \mu - \mu, \xi \rangle &= \langle \operatorname{Ad}_g^* \mu, \xi_0 + \xi_1 + \xi_2 \rangle - \langle \mu, \xi_0 + \xi_1 + \xi_2 \rangle \\ &= \langle \mu, \operatorname{Ad}_g(\xi_0 + \xi_1 + \xi_2) \rangle - \langle \mu, \xi_0 + \xi_1 + \xi_2 \rangle \\ &= \langle \mu, \xi_0 + \xi_1 + \operatorname{Ad}_g \xi_2 \rangle - \langle \mu, \xi_0 \rangle = \langle \mu, \xi_1 + \operatorname{Ad}_g \xi_2 \rangle = 0 \end{aligned}$$

because G_{q_e} acts trivially on $\mathfrak{k}_0 \oplus \mathfrak{k}_1$, $\mathfrak{k}_2$ is G_{q_e}–invariant, and $\mathfrak{m}_0 = (\mathfrak{k}_1 \oplus \mathfrak{k}_2)^\circ$. The same type of proof holds for $\mathfrak{m}_1$. □

Recall from §2.3 that $\ker \mathbb{I}(q_e) = \mathfrak{g}_{q_e} = \mathfrak{k}_0$. In particular, $\mathbb{I}(q_e)\mathfrak{k}_0 = \{0\}$. The value of $\mathbb{I}(q_e)$ on the other summands in the decomposition $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2$ is given by the following lemma.

Lemma 5.5. *For* $i \in \{1, 2\}$ *we have that* $\mathfrak{m}_i = \mathbb{I}(q_e)\mathfrak{k}_i$*.*

Proof. Let $\kappa_i \in \mathfrak{k}_i$ with $i \in \{0, 1, 2\}$ be arbitrary. Then

$$\langle \mathbb{I}(q_e)\kappa_1, \kappa_0 + \kappa_2 \rangle = \langle \mathbb{I}(q_e)\kappa_1, \kappa_0 \rangle + \langle \mathbb{I}(q_e)\kappa_1, \kappa_2 \rangle = \langle \mathbb{I}(q_e)\kappa_0, \kappa_1 \rangle + \langle \mathbb{I}(q_e)\kappa_1, \kappa_2 \rangle = 0$$

as $\ker \mathbb{I}(q_e) = \mathfrak{k}_0$ and, by Proposition 5.1 (ii), $\mathbb{I}(q_e)\mathfrak{t} \subset \mathfrak{k}_2^\circ$. This proves that $\mathbb{I}(q_e)\mathfrak{k}_1 \subset \mathfrak{m}_1$. Counting dimensions we have that $\dim \mathbb{I}(q_e)\mathfrak{k}_1 = \dim \mathfrak{k}_1 - \dim \ker (\mathbb{I}(q_e)|_{\mathfrak{k}_1}) = \dim \mathfrak{g} - \dim \mathfrak{k}_0 - \dim \mathfrak{k}_2 = \dim \mathfrak{m}_1$, since $\ker (\mathbb{I}(q_e)|_{\mathfrak{k}_1}) = \{0\}$. This proves that $\mathfrak{m}_1 = \mathbb{I}(q_e)\mathfrak{k}_1$. In an analogous way we prove the equality for $i = 2$. □

In the next paragraph we shall need the direct sum decomposition $\mathfrak{g}^* = \mathfrak{m}_1 \oplus \mathfrak{m}$, where $\mathfrak{m}_1 = \mathbb{I}(q_e)\mathfrak{t}$ and $\mathfrak{m} := \mathfrak{m}_0 \oplus \mathfrak{m}_2$. Let $\Pi_1 : \mathfrak{g}^* \to \mathbb{I}(q_e)\mathfrak{t}$ be the projection along $\mathfrak{m}$. Similarly, denote $\mathfrak{k} := \mathfrak{k}_1 \oplus \mathfrak{k}_2$, and write $\mathfrak{g} = \mathfrak{g}_{q_e} \oplus \mathfrak{k}$. Thus there is another decomposition of $\mathfrak{g}^*$, namely, $\mathfrak{g}^* = \mathfrak{g}_{q_e}^\circ \oplus \mathfrak{k}^\circ$. However, for any $\zeta \in \mathfrak{g}_{q_e}$

and any $\xi \in \mathfrak{g}$, we have $\langle \mathbb{I}(q_e)\xi, \zeta\rangle = \langle\langle \xi_Q(q_e), \zeta_Q(q_e)\rangle\rangle = 0$ since $\zeta_Q(q_e) = 0$, which shows that $\mathbb{I}(q_e)\mathfrak{g} \subset \mathfrak{g}_{q_e}^\circ$. Since $\ker \mathbb{I}(q_e) = \mathfrak{g}_{q_e}$, it follows that $\dim \mathbb{I}(q_e)\mathfrak{g} = \dim \mathfrak{g} - \dim \ker \mathbb{I}(q_e) = \dim \mathfrak{g} - \dim \mathfrak{g}_{q_e} = \dim \mathfrak{g}_{q_e}^\circ$, which shows that $\mathfrak{g}_{q_e}^\circ = \mathbb{I}(q_e)\mathfrak{g}$. Thus we also have the direct sum decomposition $\mathfrak{g}^* = \mathbb{I}(q_e)\mathfrak{g} \oplus \mathfrak{k}^\circ$. Note that $\mathbb{I}(q_e)\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, by Lemma 5.5 and that $\mathfrak{m}_0 = \mathfrak{k}^\circ$. Summarizing we have:

$$\mathfrak{g}^* = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 = \mathfrak{k}^\circ \oplus \mathbb{I}(q_e)\mathfrak{g}, \quad \text{where} \quad \mathbb{I}(q_e)\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \quad \text{and} \quad \mathfrak{m}_0 = \mathfrak{k}^\circ.$$

5.3. **The rescaled equation.** Recall that $B \subset (\mathfrak{g} \cdot q_e)^\perp$ is a G_{q_e}-invariant open neighborhood of $0_{q_e} \in (\mathfrak{g}\cdot q_e)^\perp$ such that on the open G-invariant neighborhood $G\cdot \mathrm{Exp}_{q_e}(B)$ of $G\cdot q_e$, we have $(G_{q_e}) \preceq (G_q)$ for every $q \in G \cdot \mathrm{Exp}_{q_e}(B)$. Consider the following rescaling:

$$v_{q_e} \in B \cap (T_{q_e}Q)_{\{e\}} \mapsto \tau v_{q_e} \in B \cap (T_{q_e}Q)_{\{e\}}$$
$$\mu \in \mathfrak{g}^* \mapsto \beta(\tau, \mu) \in \mathfrak{g}^*$$

where, $\tau \in I$, I is an open interval containing $[0, 1]$, and $\beta : I \times \mathfrak{g}^* \to \mathfrak{g}^*$ is chosen such that $\beta(0, \mu) = \Pi_1\mu$. So, for (v_{q_e}, μ) fixed, $(\tau v_{q_e}, \beta(\tau, \mu))$ converges to $(0_{q_e}, \Pi_1\mu)$ as $\tau \to 0$. Define

$$\beta(\tau, \mu) := \Pi_1\mu + \tau\beta'(\mu) + \tau^2\beta''(\mu)$$

for some arbitrary smooth functions $\beta', \beta'' : \mathfrak{g}^* \to \mathfrak{g}^*$. Since $\mathbb{I}$ is invertible only for points with no symmetry, we want to find conditions on β', β'' such that the expression

$$\mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))^{-1}\beta(\tau, \mu) \tag{5.1}$$

extends to a smooth function in a neighborhood of $\tau = 0$. Note that v_{q_e} is different from 0_{q_e} since $G_{v_{q_e}} = \{e\}$ by construction and $G_{0_{q_e}} = G_{q_e} \neq \{e\}$. Define

$$\Phi : I \times \left(B \cap (T_{q_e}Q)_{\{e\}}\right) \times \mathfrak{g}^* \times \mathfrak{g}_{q_e} \times \mathfrak{k} \to \mathfrak{g}^*$$

$$\Phi(\tau, v_{q_e}, \mu, \xi, \eta) := \mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))(\xi + \eta) - \beta(\tau, \mu). \tag{5.2}$$

Now we search for the velocity $\xi + \eta$ of relative equilibria among the solutions of $\Phi(\tau, v_{q_e}, \mu, \xi, \eta) = 0$. We shall prove below that ξ and η are smooth functions of τ, v_{q_e}, μ, even at $\tau = 0$. Then (5.1) shows that $\xi + \eta$ is a smooth function of τ, v_{q_e}, μ, for τ in a small neighborhood of zero.

5.4. **The Lyapunov-Schmidt procedure.** To solve $\Phi = 0$ we apply the standard Lyapunov-Schmidt method. This equation has a unique solution for $\tau \neq 0$, because $\tau v_{q_e} \in B \cap (T_{q_e}Q)_{\{e\}}$ so $\mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))$ is invertible. It remains to prove that the equation has a solution when $\tau = 0$. Denote by $D_{\mathfrak{g}_{q_e}\times\mathfrak{k}}$ the Fréchet derivative relative to the last two factors $\mathfrak{g}_{q_e} \times \mathfrak{k}$ in the definition of Φ. We have

$$\ker D_{\mathfrak{g}_{q_e}\times\mathfrak{k}}\Phi(0, v_{q_e}, \mu, \xi, \eta) = \ker \mathbb{I}(q_e) = \mathfrak{g}_{q_e}.$$

We will solve the equation $\Phi = 0$ in two steps. For this, let

$$\Pi : \mathfrak{g}^* \to \mathbb{I}(q_e)\mathfrak{g}$$

be the projection induced by the splitting $\mathfrak{g}^* = \mathbb{I}(q_e)\mathfrak{g} \oplus \mathfrak{k}^\circ$. **Step1**. Solve $\Pi \circ \Phi = 0$ for η in terms of τ, v_{q_e}, μ, ξ. For this, let

$$\widehat{\mathbb{I}}(\mathrm{Exp}_{q_e}(\tau v_{q_e})) := (\Pi \circ \mathbb{I})(\mathrm{Exp}_{q_e}(\tau v_{q_e}))|_{\mathfrak{k}} : \mathfrak{k} \to \mathbb{I}(q_e)\mathfrak{g}$$
$$\widetilde{\mathbb{I}}(\mathrm{Exp}_{q_e}(\tau v_{q_e})) := (\Pi \circ \mathbb{I})(\mathrm{Exp}_{q_e}(\tau v_{q_e}))|_{\mathfrak{g}_{q_e}} : \mathfrak{g}_{q_e} \to \mathbb{I}(q_e)\mathfrak{g}$$

where $\widehat{\mathbb{I}}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))$ is an isomorphism even when $\tau = 0$. Then we obtain

$$(\Pi \circ \Phi)(0, v_{q_e}, \mu, \xi, \eta) = \Pi[\mathbb{I}(q_e)(\xi + \eta) - \beta(0, \mu)] = \widehat{\mathbb{I}}(q_e)\eta - \Pi_1\mu. \tag{5.3}$$

Denoting $\eta_\mu := \widehat{\mathbb{I}}(q_e)^{-1}(\Pi_1\mu)$, we have $(\Pi\circ\Phi)(0, v_{q_e}, \mu, \xi, \eta_\mu) \equiv 0$. Denoting by D_η the partial Fréchet derivative relative to the variable $\eta \in \mathfrak{k}$ we get at any given point $(0, v^0_{q_e}, \mu^0, \xi^0, \eta^0)$

$$D_\eta(\Pi\circ\Phi)(0, v^0_{q_e}, \mu^0, \xi^0, \eta^0) = \widehat{\mathbb{I}}(q_e) \tag{5.4}$$

which is invertible. Thus the implicit function theorem gives a unique smooth function $\eta(\tau, v_{q_e}, \mu, \xi)$ such that $\eta(0, v^0_{q_e}, \mu^0, \xi^0) = \eta^0$ and

$$(\Pi\circ\Phi)(\tau, v_{q_e}, \mu, \xi, \eta(\tau, v_{q_e}, \mu, \xi)) \equiv 0. \tag{5.5}$$

The function η is defined in some open set in $I \times \left(B \cap (T_{q_e}Q)_{\{e\}}\right) \times \mathfrak{g}^* \times \mathfrak{g}_{q_e}$ containing $(0, v^0_{q_e}, \mu^0, \xi^0) \in \{0\} \times \left(B \cap (T_{q_e}Q)_{\{e\}}\right) \times \mathfrak{g}^* \times \mathfrak{g}_{q_e}$. If we now choose $\eta^0 = \eta_{\mu^0} = \widehat{\mathbb{I}}(q_e)^{-1}(\Pi_1\mu^0)$, then uniqueness of the solution of the implicit function theorem implies that $\eta(0, v_{q_e}, \mu, \xi) = \eta_\mu$ in the neighborhood of $(0, v^0_{q_e}, \mu^0, \xi^0)$. Later we will need the following result.

Proposition 5.6. *We have* $\eta_\mu := \widehat{\mathbb{I}}(q_e)^{-1}(\Pi_1\mu) \in \mathfrak{k}_1 \subset \mathfrak{t}$.

Proof. Since we can write $\mathfrak{t} = \ker \mathbb{I}(q_e) \oplus \mathfrak{k}_1$ we obtain

$$\widehat{\mathbb{I}}(q_e)\mathfrak{k}_1 = (\Pi\circ\mathbb{I}(q_e))\mathfrak{k}_1 = \mathbb{I}(q_e)\mathfrak{k}_1 = \mathbb{I}(q_e)(\mathfrak{t}) = \operatorname{Im}\Pi_1.$$

Now, because $\widehat{\mathbb{I}}(q_e)$ is an isomorphism, it follows that $\widehat{\mathbb{I}}(q_e)^{-1}(\Pi_1\mu) \in \mathfrak{k}_1$. □

Step2. Now we solve the equation $(Id - \Pi)\circ\Phi = 0$. For this, let

$$\varphi : I \times \left(B \cap (T_{q_e}Q)_{\{e\}}\right) \times \mathfrak{g}^* \times \mathfrak{g}_{q_e} \to \mathfrak{k}^\circ$$

$$\varphi(\tau, v_{q_e}, \mu, \xi) := (Id - \Pi)\Phi(\tau, v_{q_e}, \mu, \xi, \eta(\tau, v_{q_e}, \mu, \xi)). \tag{5.6}$$

In particular, $\varphi(0, v_{q_e}, \mu, \xi) = (Id-\Pi)(\mathbb{I}(q_e)(\xi+\eta_\mu) - \Pi_1\mu)$. Since $\operatorname{Im}\mathbb{I}(q_e) = \operatorname{Im}\Pi$ and $\operatorname{Im}\Pi_1 = \mathbb{I}(q_e)\mathfrak{t} \subset \mathbb{I}(q_e)\mathfrak{g}$, it follows that $\varphi(0, v_{q_e}, \mu, \xi) \equiv 0$. We shall solve for $\xi \in \mathfrak{g}_{q_e}$, in the neighborhood of $(0, v^0_{q_e}, \mu^0, \xi^0)$ found in Step 1, the equation $\varphi(\tau, v_{q_e}, \mu, \xi) = 0$. To do this, we shall need information about the higher derivatives of φ with respect to τ, evaluated at $\tau = 0$.

Lemma 5.7. *Let ξ, $\eta \in \mathfrak{g}$ and $q \in Q$. Suppose that $\mathbf{d}V_\eta(q) = 0$, where V_η is the augmented potential and suppose that both ξ and $[\xi, \eta]$ belong to $\mathfrak{g}_q$. Then $\mathbf{d}\langle\mathbb{I}(\cdot)\xi, \eta\rangle(q) = 0$.*

Proof. Since $\mathbf{d}V_\eta(q) = 0$, $\eta_Q(q)$ is a relative equilibrium by Proposition 3.3, that is, $X_H(\alpha_q) = \eta_{T^*Q}(\alpha_q)$, where $\alpha_q = \mathbb{F}L(\eta_Q(q))$. Now suppose that both ξ, $[\xi, \eta] \in \mathfrak{g}_q$. Then

$$\xi_{T^*Q}(\alpha_q) = \left.\frac{d}{dt}\right|_{t=0} \mathbb{F}L(\exp(t\xi)\cdot\eta_Q(q)) = \mathbb{F}L([\xi, \eta]_Q(q)) = 0,$$

where we have used that $g \cdot \eta_Q(q) = (\operatorname{Ad}_g \eta)_Q(g \cdot q)$. It follows that $(\eta + \xi)_{T^*Q}(\alpha_q) = X_H(\alpha_q)$ and hence, again by Proposition 3.3, that $0 = \mathbf{d}V_{\eta+\xi}(q) = \mathbf{d}V_\eta(q) - \mathbf{d}\langle\mathbb{I}(\cdot)\eta, \xi\rangle(q) - \frac{1}{2}\mathbf{d}\|\xi_Q(\cdot)\|^2(q)$. However, $\mathbf{d}\|\xi_Q(\cdot)\|^2(q) = 0$ since $\xi \in \mathfrak{g}_q$, as an easy coordinate computation shows. Since $\mathbf{d}V_\eta(q) = 0$ by hypothesis, we have $\mathbf{d}\langle\mathbb{I}(\cdot)\eta, \xi\rangle(q) = 0$. Symmetry of $\mathbb{I}(q)$ proves the result. □

Let now $\xi \in \mathfrak{g}_{q_e}$ and $\eta \in \mathfrak{t}$. Since $\mathfrak{g}_{q_e} \subset \mathfrak{t}$, we have $[\xi, \eta] = 0 \in \mathfrak{g}_{q_e}$. In addition, hypothesis **(H)** and Proposition 3.3, guarantee that $\mathbf{d}V_\xi(q_e) = 0$ which shows that all hypotheses of the previous lemma are satisfied. Therefore,

$$\mathbf{d}\langle\mathbb{I}(\cdot)\xi, \eta\rangle(q_e) = 0 \quad \text{for} \quad \xi \in \mathfrak{g}_{q_e},\ \eta \in \mathfrak{t}. \tag{5.7}$$

5.5. **The bifurcation equation.** Now we can proceed with the study of equation $\varphi = (Id - \Pi) \circ \Phi = 0$. We have

$$\frac{\partial \varphi}{\partial \tau}(\tau, v_{q_e}, \mu, \xi) = (Id - \Pi) \left[T_{\tau v_{q_e}}(\mathbb{I} \circ \mathrm{Exp}_{q_e})(v_{q_e})(\xi + \eta(\tau, v_{q_e}, \mu, \xi)) + \mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e})) \frac{\partial \eta}{\partial \tau}(\tau, v_{q_e}, \mu, \xi) \right.$$
$$\left. - \frac{\partial \beta}{\partial \tau}(\tau, \mu) \right]. \tag{5.8}$$

Proposition 5.8. $\frac{\partial}{\partial \tau}\varphi(0, v_{q_e}, \mu, \xi) \equiv -(Id - \Pi)\beta'(\mu)$.

Proof. Formula (5.8) gives for $\tau = 0$

$$\frac{\partial \varphi}{\partial \tau}(0, v_{q_e}, \mu, \xi) = (Id - \Pi) \left[\left(T_{q_e}\mathbb{I}(v_{q_e}) \right)(\xi + \eta_\mu) + \mathbb{I}(q_e) \frac{\partial \eta}{\partial \tau}(0, v_{q_e}, \mu, \xi) - \frac{\partial \beta}{\partial \tau}(0, \mu) \right].$$

Now, because $\mathrm{Im}\, \mathbb{I}(q_e) = \mathrm{Im}\, \Pi$ we obtain $(Id - \Pi) \circ \mathbb{I}(q_e) = 0$ and hence the second summand vanishes. From (5.7) we have that $(T_{q_e}\mathbb{I}(v_{q_e}))(\mathfrak{t}) \subset \mathfrak{g}_{q_e}^\circ = \mathrm{Im}\, \Pi$. Using Proposition 5.6 and since $\xi \in \mathfrak{g}_{q_e} \subset \mathfrak{t}$, we obtain that $\xi + \eta_\mu \in \mathfrak{t}$. Therefore $(Id - \Pi)[(T_{q_e}\mathbb{I}(v_{q_e}))(\xi + \eta_\mu)] = 0$. Since $\frac{\partial \beta}{\partial \tau}(0, \mu) = \beta'(\mu)$, we obtain the desired equality. $\square$

Let us impose the additional condition $\beta'(\mu) \subset \mathrm{Im}\, \Pi$. Then it follows that

$$\varphi(\tau, v_{q_e}, \mu, \xi) = \tau^2 \psi(\tau, v_{q_e}, \mu, \xi).$$

for some smooth function ψ where

$$\psi(0, v_{q_e}, \mu, \xi) = \frac{1}{2} \frac{\partial^2 \varphi}{\partial \tau^2}(0, v_{q_e}, \mu, \xi)$$

We begin by solving the equation

$$\psi(0, v_{q_e}, \mu, \xi) = 0$$

for ξ as a function of v_{q_e} and μ. Equivalently, we have to solve

$$\frac{1}{2} \frac{\partial^2 \varphi}{\partial \tau^2}(0, v_{q_e}, \mu, \xi) = 0.$$

To compute this second derivative of φ we shall use (5.8). We begin by noting that $\tau \in I \mapsto T_{\tau v_{q_e}}(\mathbb{I} \circ \mathrm{Exp}_{q_e})(v_{q_e})$ is a smooth path in $L(\mathfrak{g}, \mathfrak{g}^*)$ and so we can define the linear operator from $\mathfrak{g}$ to $\mathfrak{g}^*$ by

$$A_{v_{q_e}} := \left. \frac{\partial}{\partial \tau} \right|_{\tau=0} T_{\tau v_{q_e}}(\mathbb{I} \circ \mathrm{Exp}_{q_e})(v_{q_e}) \in L(\mathfrak{g}, \mathfrak{g}^*).$$

With this notation, formulas (5.8), (5.2), (5.6), and Proposition 5.6 yield

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial \tau^2}(0, v_{q_e}, \mu, \xi) &= (Id - \Pi) \left[A_{v_{q_e}}(\xi + \eta_\mu) + 2 T_{q_e}\mathbb{I}(v_{q_e}) \frac{\partial \eta}{\partial \tau}(0, v_{q_e}, \mu, \xi) \right. \\ &\qquad \left. + \mathbb{I}(q_e) \frac{\partial^2 \eta}{\partial \tau^2}(0, v_{q_e}, \mu, \xi) - 2\beta''(\mu) \right] \\ &= (Id - \Pi) \left[A_{v_{q_e}}(\xi + \eta_\mu) + 2 T_{q_e}\mathbb{I}(v_{q_e}) \frac{\partial \eta}{\partial \tau}(0, v_{q_e}, \mu, \xi) - 2\beta''(\mu) \right] \end{aligned} \tag{5.9}$$

since $(Id - \Pi)\mathbb{I}(q_e) \frac{\partial^2 \eta}{\partial \tau^2}(0, v_{q_e}, \mu, \xi) = 0$. Let $\{\xi_1, ..., \xi_p\}$ be a basis of $\mathfrak{g}_{q_e}$. Since $\partial^2 \varphi(\tau, v_{q_e}, \mu, \xi)/\partial \tau^2 \in \mathfrak{k}^\circ$ and $\mathfrak{g} = \mathfrak{g}_{q_e} \oplus \mathfrak{k}$ the equation $\partial^2 \varphi(0, v_{q_e}, \mu, \xi)/\partial \tau^2 = 0$ is equivalent to the following system of p equations

$$\left\langle \frac{\partial^2 \varphi}{\partial \tau^2}(0, v_{q_e}, \mu, \xi), \xi_b \right\rangle = 0, \quad \text{for all} \quad b = 1, \ldots, p,$$

which, by (5.9), is

$$\left\langle (Id-\Pi)\left[A_{v_{q_e}}(\xi+\eta_\mu)+2T_{q_e}\mathbb{I}(v_{q_e})\frac{\partial\eta}{\partial\tau}(0,v_{q_e},\mu,\xi)-2\beta''(\mu)\right],\xi_b\right\rangle=0,\quad \text{for all}\quad b=1,\ldots,p.$$

We shall show that in this expression we can drop the projector $Id-\Pi$. Indeed, let $\alpha=\alpha_0+\alpha_1+\alpha_2\in\mathfrak{g}^*=\mathfrak{m}_0\oplus\mathfrak{m}_1\oplus\mathfrak{m}_2$, where $\alpha_i\in\mathfrak{m}_i$, for $i=0,1,2$. Since $\Pi:\mathfrak{g}^*\to\mathbb{I}(q_e)\mathfrak{g}=\mathfrak{m}_1\oplus\mathfrak{m}_2$, we have

$$\langle (Id-\Pi)\alpha,\xi_b\rangle=\langle\alpha,\xi_b\rangle-\langle\alpha_1,\xi_b\rangle-\langle\alpha_2,\xi_b\rangle=\langle\alpha,\xi_b\rangle$$

because $\langle\alpha_1,\xi_b\rangle=0$, since $\alpha_1\in\mathfrak{m}_1=(\mathfrak{k}_0\oplus\mathfrak{k}_2)^\circ$, $\xi_b\in\mathfrak{g}_{q_e}=\mathfrak{k}_0$, and $\langle\alpha_2,\xi_b\rangle=0$, since $\alpha_2\in\mathfrak{m}_2=(\mathfrak{k}_0\oplus\mathfrak{k}_1)^\circ$, $\xi_b\in\mathfrak{g}_{q_e}=\mathfrak{k}_0$. The system to be solved is hence

$$\left\langle A_{v_{q_e}}(\xi+\eta_\mu)+2T_{q_e}\mathbb{I}(v_{q_e})\frac{\partial\eta}{\partial\tau}(0,v_{q_e},\mu,\xi)-2\beta''(\mu),\xi_b\right\rangle=0,\quad \text{for all}\quad b=1,\ldots,p. \tag{5.10}$$

In what follows we need the expression for $\frac{\partial\eta}{\partial\tau}(0,v_{q_e},\mu,\xi)$. Differentiating (5.5) relative to τ at zero and taking into account (5.4) and (5.2), we get

$$\begin{aligned}\frac{\partial\eta}{\partial\tau}(0,v_{q_e},\mu,\xi)&=-\widehat{\mathbb{I}}(q_e)^{-1}\frac{\partial}{\partial\tau}(\Pi\circ\Phi)(0,v_{q_e},\mu,\xi,\eta_\mu)\\&=-\widehat{\mathbb{I}}(q_e)^{-1}\Pi\left[T_{q_e}\mathbb{I}(v_{q_e})(\xi+\eta_\mu)-\beta'(\mu)\right]\\&=-\left(\widehat{\mathbb{I}}(q_e)^{-1}\circ T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\right)\xi-\left(\widehat{\mathbb{I}}(q_e)^{-1}\circ T_{q_e}\widehat{\mathbb{I}}(v_{q_e})\circ\widehat{\mathbb{I}}(q_e)^{-1}\right)(\Pi_1\mu)+\widehat{\mathbb{I}}(q_e)^{-1}(\beta'(\mu))\end{aligned}\tag{5.11}$$

since $T_{q_e}\widetilde{\mathbb{I}}=\Pi\circ T_{q_e}\mathbb{I}|_{\mathfrak{g}_{q_e}}$ and $T_{q_e}\widehat{\mathbb{I}}=\Pi\circ T_{q_e}\mathbb{I}|_{\mathfrak{k}}$. Expanding ξ in the basis $\{\xi_1,\ldots,\xi_p\}$ as $\xi=\alpha^i\xi_i$ and taking into account the above expression, the system (5.10) is equivalent to the following system of linear equations in the unknowns $\alpha^1,\ldots,\alpha^p$

$$A_{ab}\alpha^a+B_b=0,\quad a,b=1,\ldots,p,$$

where

$$A_{ab}:=\langle A_{v_{q_e}}\xi_a,\xi_b\rangle-2\left\langle\left(T_{q_e}\mathbb{I}(v_{q_e})\circ\widehat{\mathbb{I}}(q_e)^{-1}\circ T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\right)\xi_a,\xi_b\right\rangle\tag{5.12}$$

$$\begin{aligned}B_b:=&\left\langle\left(A_{v_{q_e}}\circ\widehat{\mathbb{I}}(q_e)^{-1}\circ\Pi_1\right)\mu,\xi_b\right\rangle-2\left\langle\left(T_{q_e}\mathbb{I}(v_{q_e})\circ\widehat{\mathbb{I}}(q_e)^{-1}\circ T_{q_e}\widehat{\mathbb{I}}(v_{q_e})\circ\widehat{\mathbb{I}}(q_e)^{-1}\circ\Pi_1\right)\mu,\xi_b\right\rangle\\&+2\left\langle\left(T_{q_e}\mathbb{I}(v_{q_e})\circ\widehat{\mathbb{I}}(q_e)^{-1}\right)\beta'(\mu),\xi_b\right\rangle-\langle\beta''(\mu),\xi_b\rangle.\end{aligned}\tag{5.13}$$

Denote by $A:=[A_{ab}]$ the $p\times p$ matrix with entries A_{ab}. Thus, if $v_{q_e}\notin\mathcal{Z}_\mu=:\{v_{q_e}\in B\cap(T_{q_e}Q)_{\{e\}}\mid\det A=0\}$ this linear system has a unique solution for $\alpha^1,\ldots,\alpha^p$, that is for ξ, as function of v_{q_e}, μ. we shall denote this solution by $\xi_0(v_{q_e},\mu)$. *Summarizing, if $v_{q_e}\notin\mathcal{Z}_\mu$, then $\xi_0(v_{q_e},\mu)$ is the unique solution of the equation*

$$\frac{\partial^2\varphi}{\partial\tau^2}(0,v_{q_e},\mu,\xi)=0.\tag{5.14}$$

Lemma 5.9. *The set $\mathcal{Z}_\mu$ is closed and G_{q_e}–invariant in $B\cap(T_{q_e}Q)_{\{e\}}$.*

Proof. The set $\mathcal{Z}_\mu$ is obviously closed. Since $\mathfrak{k}$ is G_{q_e}–invariant it follows that $\mathfrak{k}^\circ$ is G_{q_e}–invariant. Formula (2.1) shows that $\mathbb{I}(q_e)\mathfrak{g}$ is also G_{q_e}–invariant. Thus the direct sum $\mathbb{I}(q_e)\mathfrak{g}\oplus\mathfrak{k}^\circ$ is a G_{q_e}–invariant decomposition of $\mathfrak{g}^*$ and therefore $\Pi:\mathfrak{g}^*\to\mathbb{I}(q_e)\mathfrak{g}$ is G_{q_e}–equivariant. From the G_{q_e}–equivariance of

Exp_{q_e} and (2.1), it follows that $\mathbb{I}(\mathrm{Exp}_{q_e}(h\cdot v_{q_e})) = \mathrm{Ad}^*_{h^{-1}} \circ \mathbb{I}(\mathrm{Exp}_{q_e}(v_{q_e}))\circ \mathrm{Ad}_{h^{-1}} = \mathrm{Ad}^*_{h^{-1}} \circ \mathbb{I}(\mathrm{Exp}_{q_e}(v_{q_e}))$ for any $h \in G_{q_e}$ since $G_{q_e} \subset \mathbb{T}$ and is therefore Abelian. Thus

$$\begin{aligned}\widetilde{\mathbb{I}}(\mathrm{Exp}_{q_e}(h\cdot v_{q_e})) &= \Pi \circ \mathbb{I}(\mathrm{Exp}_{q_e}(T_{q_e}\Psi_h \cdot v_{q_e}))|_{\mathfrak{g}_{q_e}} = \Pi \circ \mathrm{Ad}^*_{h^{-1}} \circ \mathbb{I}(\mathrm{Exp}_{q_e}(v_{q_e}))|_{\mathfrak{g}_{q_e}} \\ &= \mathrm{Ad}^*_{h^{-1}} \circ \Pi \circ \mathbb{I}(\mathrm{Exp}_{q_e}(v_{q_e}))|_{\mathfrak{g}_{q_e}} = \mathrm{Ad}^*_{h^{-1}} \circ \widetilde{\mathbb{I}}(\mathrm{Exp}_{q_e}(v_{q_e}))\end{aligned}$$

for all $h \in G_{q_e}$ and $v_{q_e} \in B$. Replacing here v_{q_e} by sv_{q_e} and taking the s–derivative at zero, shows that $T_{q_e}\widetilde{\mathbb{I}}(h\cdot v_{q_e})\xi = \mathrm{Ad}^*_{h^{-1}}\left(T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\xi\right)$ for any $h \in G_{q_e}$ and $\xi \in \mathfrak{g}_{q_e}$, that is, $T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\xi$ is G_{q_e}–equivariant as a function of v_{q_e}, for all $\xi \in \mathfrak{g}_{q_e}$. Similarly $T_{q_e}\mathbb{I}(h\cdot v_{q_e}) = \mathrm{Ad}^*_{h^{-1}} \circ T_{q_e}\mathbb{I}(v_{q_e}) \circ \mathrm{Ad}_{h^{-1}}$. From (2.1) and the definition of $\widehat{\mathbb{I}}(q_e)^{-1}$, it follows that $\widehat{\mathbb{I}}(q_e)^{-1} = \mathrm{Ad}_h \circ \widehat{\mathbb{I}}(q_e)^{-1} \circ \mathrm{Ad}^*_h$ for any $h \in G_{q_e}$. Thus, for $h \in G_{q_e}$, the second summand in A_{ab} becomes

$$\begin{aligned}&\left\langle \left(T_{q_e}\mathbb{I}(h\cdot v_{q_e}) \circ \widehat{\mathbb{I}}(q_e)^{-1} \circ T_{q_e}\widetilde{\mathbb{I}}(h\cdot v_{q_e})\right)\xi_a, \xi_b\right\rangle \\ &= \left\langle \left(\mathrm{Ad}^*_{h^{-1}} \circ T_{q_e}\mathbb{I}(v_{q_e}) \circ \mathrm{Ad}_{h^{-1}} \circ \widehat{\mathbb{I}}(q_e)^{-1} \circ \mathrm{Ad}^*_{h^{-1}} \circ T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\right)\xi_a, \xi_b\right\rangle \\ &= \left\langle \left(\mathrm{Ad}^*_{h^{-1}} \circ T_{q_e}\mathbb{I}(v_{q_e}) \circ \widehat{\mathbb{I}}(q_e)^{-1} \circ T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\right)\xi_a, \xi_b\right\rangle \\ &= \left\langle \left(T_{q_e}\mathbb{I}(v_{q_e}) \circ \widehat{\mathbb{I}}(q_e)^{-1} \circ T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\right)\xi_a, \mathrm{Ad}_{h^{-1}}\xi_b\right\rangle \\ &= \left\langle \left(T_{q_e}\mathbb{I}(v_{q_e}) \circ \widehat{\mathbb{I}}(q_e)^{-1} \circ T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\right)\xi_a, \xi_b\right\rangle\end{aligned}$$

since $\mathrm{Ad}_{h^{-1}}\xi_b = 0$ because $h \in G_{q_e}$ and $\xi_b \in \mathfrak{g}_{q_e}$. This shows that the second summand in A_{ab} is G_{q_e}–invariant. Next, we show that the first summand in A_{ab} is G_{q_e}– invariant. To see this note that

$$\left\langle A_{v_{q_e}}\xi_a, \xi_b\right\rangle = \left.\frac{\partial}{\partial\tau}\right|_{\tau=0}\left\langle T_{\tau v_{q_e}}(\mathbb{I}\circ \mathrm{Exp}_{q_e})(v_{q_e})\xi_a, \xi_b\right\rangle = \left.\frac{\partial^2}{\partial\tau^2}\right|_{\tau=0}\left\langle \mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))\xi_a, \xi_b\right\rangle.$$

Therefore, for any $h \in G_{q_e}$ we get from (2.1)

$$\begin{aligned}\left\langle A_{h\cdot v_{q_e}}\xi_a, \xi_b\right\rangle &= \left.\frac{\partial^2}{\partial\tau^2}\right|_{\tau=0}\left\langle \mathbb{I}(\mathrm{Exp}_{q_e}(\tau h\cdot v_{q_e}))\xi_a, \xi_b\right\rangle = \left.\frac{\partial^2}{\partial\tau^2}\right|_{\tau=0}\left\langle \mathbb{I}(h\cdot \mathrm{Exp}_{q_e}(\tau v_{q_e}))\xi_a, \xi_b\right\rangle \\ &= \left.\frac{\partial^2}{\partial\tau^2}\right|_{\tau=0}\left\langle \mathrm{Ad}^*_{h^{-1}}\,\mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))\,\mathrm{Ad}_{h^{-1}}\xi_a, \xi_b\right\rangle \\ &= \left.\frac{\partial^2}{\partial\tau^2}\right|_{\tau=0}\left\langle \mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))\,\mathrm{Ad}_{h^{-1}}\xi_a, \mathrm{Ad}_{h^{-1}}\xi_b\right\rangle \\ &= \left.\frac{\partial^2}{\partial\tau^2}\right|_{\tau=0}\left\langle \mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))\xi_a, \xi_b\right\rangle = \left\langle A_{v_{q_e}}\xi_a, \xi_b\right\rangle,\end{aligned}$$

as required. □

Proposition 5.10. *The equation* $\varphi(\tau, v_{q_e}, \mu, \xi) = 0$ *for* $(\tau, v_{q_e}, \mu, \xi) \in I \times \left(B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu\right) \times \mathfrak{g}^* \times \mathfrak{g}_{q_e}$ *has a unique smooth solution* $\xi(\tau, v_{q_e}, \mu) \in \mathfrak{g}_{q_e}$ *for* $(\tau, v_{q_e}, \mu) \in I \times (B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu) \times \mathfrak{g}^*$.

Proof. Denote by D_ξ the Fréchet derivative relative to the variable $\xi \in \mathfrak{g}_{q_e}$. Recall that $\xi_0(v_{q_e}, \mu) \in \mathfrak{g}_{q_e}$ is the unique solution of the equation $\frac{\partial^2 \varphi}{\partial \tau^2}(0, v_{q_e}, \mu, \xi) = 0$. Formulas (5.9) and (5.11) yield

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial \tau^2}(0, v_{q_e}, \mu, \xi) = (Id - \Pi) \Big[& A_{v_{q_e}}(\xi + \eta_\mu) - 2 \left(T_{q_e}\mathbb{I}(v_{q_e}) \circ \widehat{\mathbb{I}}(q_e)^{-1} \circ T_{q_e} \widetilde{\mathbb{I}}(v_{q_e}) \right) \xi \\ & - 2 \left(T_{q_e}\mathbb{I}(v_{q_e}) \circ \widehat{\mathbb{I}}(q_e)^{-1} \circ T_{q_e} \widehat{\mathbb{I}}(v_{q_e}) \circ \widehat{\mathbb{I}}(q_e)^{-1} \right) (\Pi_1 \mu) \\ & + 2 \left(T_{q_e}\mathbb{I}(v_{q_e}) \circ \widehat{\mathbb{I}}(q_e)^{-1} \right) (\beta'(\mu)) - 2\beta''(\mu) \Big] \end{aligned} \tag{5.15}$$

and hence

$$D_\xi \frac{\partial^2 \varphi}{\partial \tau^2}(0, v_{q_e}, \mu, \xi_0(v_{q_e}, \mu)) = (Id - \Pi) \left[A_{v_{q_e}}|_{\mathfrak{g}_{q_e}} - 2T_{q_e}\mathbb{I}(v_{q_e}) \circ \widehat{\mathbb{I}}(q_e)^{-1} \circ T_{q_e} \widetilde{\mathbb{I}}(v_{q_e}) \right] : \mathfrak{g}_{q_e} \to \mathfrak{k}^\circ.$$

We shall prove that this linear map is injective. To see this, note that relative to the basis $\{\xi_1, \ldots, \xi_p\}$ of $\mathfrak{g}_{q_e}$ this linear operator has matrix A by (5.12). Thus, if $v_{q_e} \notin \mathcal{Z}_\mu$, this matrix is invertible. In particular, this linear operator is injective.

Since $\mathfrak{g} = \mathfrak{g}_{q_e} \oplus \mathfrak{k}$, it follows that $\dim \mathfrak{g}_{q_e} = \dim \mathfrak{g} - \dim \mathfrak{k} = \dim \mathfrak{k}^\circ$, so the injectivity of the map $D_\xi \frac{\partial^2 \varphi}{\partial \tau^2}(0, v^0_{q_e}, \mu^0, \xi_0(v^0_{q_e}, \mu^0))$ implies that it is an isomorphism. Therefore, if $v_{q_e} \in B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu$ is near $v^0_{q_e}$, the implicit function theorem, guarantees the existence of an open neighborhood $V_0 \subset I \times (B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu) \times \mathfrak{g}^*$ containing $(0, v^0_{q_e}, \mu^0) \in \{0\} \times (B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu) \times \mathfrak{g}^*$ and of a unique smooth function $\xi : V_0 \to \mathfrak{g}_{q_e}$ satisfying $\varphi(\tau, v_{q_e}, \mu, \xi(\tau, v_{q_e}, \mu)) = 0$ such that $\xi(0, v^0_{q_e}, \mu^0) = \xi_0(v^0_{q_e}, \mu^0)$. On the other hand, for $\tau \neq 0$, the equation $\varphi(\tau, v_{q_e}, \mu, \cdot) = 0$ has a unique solution for ξ, namely the $\mathfrak{g}_{q_e}$-component of $\mathbb{I}(\mathrm{Exp}_{q_q}(\tau v_{q_e}))^{-1}\beta(\tau, \mu)$, which is a smooth function of τ, v_{q_e}, μ. This is true since $\xi + \eta = \mathbb{I}(\mathrm{Exp}_{q_q}(\tau v_{q_e}))^{-1}\beta(\tau, \mu)$ by construction and we determined the two components $\xi \in \mathfrak{g}_{q_e}$ and $\eta \in \mathfrak{k}$ in $\mathfrak{g} = \mathfrak{g}_{q_e} \oplus \mathfrak{k}$ via the Lyapunov-Schmidt method, precisely in order that this equality be satisfied. Therefore, the solution $\xi(\tau, v_{q_e}, \mu)$ obtained above by the implicit function theorem must coincide with the $\mathfrak{g}_{q_e}$-component of $\mathbb{I}(\mathrm{Exp}_{q_q}(\tau v_{q_e}))^{-1}\beta(\tau, \mu)$ for $\tau > 0$. Since this entire argument involving the Lyapunov-Schmidt procedure was carried out for any $(v^0_{q_e}, \mu^0)$, it follows that the equation $\varphi(\tau, v_{q_e}, \mu, \xi) = 0$ has a unique smooth solution $\xi(\tau, v_{q_e}, \mu) \in \mathfrak{g}_{q_e}$ for $(\tau, v_{q_e}, \mu) \in I \times (B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu) \times \mathfrak{g}^*$. □

Remark 5.11. The previous proposition says that if we define

$$\zeta(\tau, v_{q_e}, \mu) = \mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))^{-1}\beta(\tau, \mu)$$

on $(I \setminus \{0\}) \times (B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu) \times \mathfrak{g}^*$, then $\zeta(\tau, v_{q_e}, \mu)$ can be smoothly extended for $\tau = 0$. We have, in fact, $\zeta(\tau, v_{q_e}, \mu) = \xi(\tau, v_{q_e}, \mu) + \eta(\tau, v_{q_e}, \mu, \xi(\tau, v_{q_e}, \mu))$, where $\eta(\tau, v_{q_e}, \mu, \xi)$ was found in the first step of the Lyapunov-Schmidt procedure and $\xi(\tau, v_{q_e}, \mu)$ in the second step, as given in Proposition 5.10. Note also that $\zeta(0, v_{q_e}, \mu) = \xi_0(v_{q_e}, \mu) + \widehat{\mathbb{I}}(q_e)^{-1}\Pi_1\mu \in \mathfrak{t}$.

5.6. **A simplified version of the amended potential criterion.** At this point we have a candidate for a bifurcating branch from the set of relative equilibria $\mathfrak{t} \cdot q_e$. This branch will start at $\zeta(0, v_{q_e}, \mu)_Q(q_e) \in \mathfrak{t} \cdot q_e \subset T_{q_e}Q$. By Lemma 5.2, the isotropy subgroup of $\zeta(0, v_{q_e}, \mu)_Q(q_e)$ equals G_{q_e}, for any $v_{q_e} \in B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu$ and $\mu \in \mathfrak{g}^*$. The isotropy groups of the points on the curve $\zeta(\tau, v_{q_e}, \mu)_Q(\mathrm{Exp}_{q_e}(\tau v_{q_e}))$, for $\tau \neq 0$, are all trivial, by construction. Hence $\zeta(\tau, v_{q_e}, \mu)_Q(\mathrm{Exp}_{q_e}(\tau v_{q_e}))$ is a curve that has the properties of the bifurcating branch of relative equilibria with broken symmetry that we are looking for. We do not know yet that all points on this curve are in fact relative equilibria. Thus, we shall search for conditions on v_{q_e} and μ that guarantee that each point on the curve

$\tau \mapsto \zeta(\tau, v_{q_e}, \mu)_Q(\mathrm{Exp}_{q_e}(\tau v_{q_e}))$ is a relative equilibrium. This will be done by using the amended potential criterion (see Proposition 3.4) which is applicable because all base points of this curve, namely $\mathrm{Exp}_{q_e}(\tau v_{q_e})$, have trivial isotropy for $\tau \neq 0$. To carry this out, we need some additional geometric information. From standard theory of proper Lie group actions (see e.g. [4], §2.3, or [7]) it follows that

the map

$$[v_{q_e}, \mu]_{G_{q_e}} \in (B \times \mathfrak{g}^*)/G_{q_e} \longmapsto [\mathrm{Exp}_{q_e}(v_{q_e}), \mu]_G \in ((G \cdot \mathrm{Exp}_{q_e} B) \times \mathfrak{g}^*)/G \tag{5.16}$$

is a homeomorphism of $(B \times \mathfrak{g}^*)/G_{q_e}$ with $((G \cdot \mathrm{Exp}_{q_e} B) \times \mathfrak{g}^*)/G$ and that its restriction to $((B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu) \times \mathfrak{g}^*)/G_{q_e}$ is a diffeomorphism onto its image. We think of a pair $(\mathrm{Exp}_{q_e}(v_{q_e}), \mu)$ as the base point of a relative equilibrium and its momentum value. All these relative equilibria come in G-orbits. The homeomorphism (5.16) allows the identification of G-orbits of relative equilibria with G_{q_e}-orbits of certain pairs (v_{q_e}, μ). We shall work in what follows on both sides of this identification, based on convenience. We will need the following lemma, which is a special case of stability of the transversality of smooth maps (see e.g. [5]).

Lemma 5.12. *Let G be a Lie group acting on a Riemannian manifold Q, $q \in Q$, and let $\mathfrak{k} \subset \mathfrak{g}$ be a subspace satisfying $\mathfrak{k} \cap \mathfrak{g}_q = \{0\}$. Let $V \subset T_qQ$ be a subspace such that $\mathfrak{k} \cdot q \oplus V = T_qQ$. Then there is an $\epsilon > 0$ such that if $\|v_q\| < \epsilon$,*

$$T_{\mathrm{Exp}_q(v_q)}Q = \mathfrak{k} \cdot \mathrm{Exp}_q(v_q) \oplus (T_{v_q} \mathrm{Exp}_q)V.$$

To deal with G-orbits of relative equilibria, we need a different splitting of the same nature. The following result is modeled on a proposition in [6].

Proposition 5.13. *Let $v_{q_e} \in B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu$ be given. Consider the principal G_{q_e}-bundle $B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu \to [B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu]/G_{q_e}$ (this is implied by Lemma 5.9). Let $\widetilde{U}$ be a neighborhood of $[0_{q_e}] \in (T_{q_e}Q)/G_{q_e}$ and define the open set $U := \widetilde{U} \cap [B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu]/G_{q_e}$ in $[B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu]/G_{q_e}$. Let $\sigma : U \subset [B \cap (T_{q_e}Q)_{\{e\}}) \setminus \mathcal{Z}_\mu]/G_{q_e} \to B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu$ be a smooth section, $[v_{q_e}] \in U$, and $\overline{\sigma} := \mathrm{Exp}_{q_e} \circ \sigma : U \to Q$. Then there exists $\epsilon > 0$ such that for $0 < \tau < \epsilon$ sufficiently small, we have*

$$T_{\overline{\sigma}([\tau v_{q_e}])}Q = \mathfrak{t} \cdot \overline{\sigma}([\tau v_{q_e}]) \oplus T_{[\tau v_{q_e}]}\overline{\sigma}(T_{[\tau v_{q_e}]}U) \oplus (T_{\sigma([\tau v_{q_e}])} \mathrm{Exp}_{q_e})(\mathfrak{k}_2 \cdot q_e).$$

Proof. Since $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{k}_0 = \mathfrak{g}_{q_e}$ we have $T_{q_e}Q = \mathfrak{k}_1 \cdot q_e \oplus \mathfrak{k}_2 \cdot q_e \oplus (\mathfrak{g} \cdot q_e)^\perp$. Apply the above lemma with $\mathfrak{k} = \mathfrak{k}_1$ and $V = \mathfrak{k}_2 \cdot q_e \oplus (\mathfrak{g} \cdot q_e)^\perp$. For the $\epsilon > 0$ in the statement choose τ such that $0 < \tau < \epsilon$ and $\|\sigma([\tau v_{q_e}])\| < \epsilon$. Then

$$\begin{aligned} T_{\overline{\sigma}([\tau v_{q_e}])}Q &= \mathfrak{k}_1 \cdot \overline{\sigma}([\tau v_{q_e}]) \oplus (T_{\sigma([\tau v_{q_e}])} \mathrm{Exp}_{q_e})(\mathfrak{k}_2 \cdot q_e \oplus (\mathfrak{g} \cdot q_e)^\perp) \\ &= \mathfrak{k}_1 \cdot \overline{\sigma}([\tau v_{q_e}]) \oplus (T_{\sigma([\tau v_{q_e}])} \mathrm{Exp}_{q_e})((\mathfrak{g} \cdot q_e)^\perp) \oplus (T_{\sigma([\tau v_{q_e}])} \mathrm{Exp}_{q_e})(\mathfrak{k}_2 \cdot q_e). \end{aligned} \tag{5.17}$$

since Exp_{q_e} is a diffeomorphism on $B \subset (\mathfrak{g} \cdot q_e)^\perp$. Since (σ, U) is a smooth local section, $\mathcal{Z}_\mu$ is closed and G_{q_e}-invariant in $B \cap (T_{q_e}Q)_{\{e\}}$, and $(T_{q_e}Q)_{\{e\}}$ is open in $T_{q_e}Q$, it follows that $B \cap (T_{q_e}Q)_{\{e\}}$ is open in $(\mathfrak{g} \cdot q_e)^\perp$ and thus we get

$$(\mathfrak{g} \cdot q_e)^\perp = T_{\sigma([\tau v_{q_e}])}(B \cap (T_{q_e}Q)_{\{e\}} \setminus \mathcal{Z}_\mu) = T_{[\tau v_{q_e}]}\sigma(T_{[\tau v_{q_e}]}U) \oplus \mathfrak{k}_0 \cdot \sigma([\tau v_{q_e}]),$$

where $\mathfrak{k}_0 \cdot \sigma([\tau v_{q_e}]) = \{\zeta_{T_{q_e}Q}(\sigma([\tau v_{q_e}]) \mid \zeta \in \mathfrak{k}_0\}$. The G_{q_e}-equivariance of Exp_{q_e} implies that

$$T_{u_{q_e}} \mathrm{Exp}_{q_e}(\xi_{T_{q_e}Q}(u_{q_e})) = \xi_Q(\mathrm{Exp}_{q_e}(u_{q_e})) \quad \text{for all} \quad \xi \in \mathfrak{k}_0, \quad u_{q_e} \in T_{q_e}Q$$

and hence

$$
\begin{aligned}
(5.18) \qquad & (T_{\sigma([\tau v_{q_e}])}\operatorname{Exp}_{q_e})((\mathfrak{g}\cdot q_e)^{\perp}) \\
&= (T_{\sigma([\tau v_{q_e}])}\operatorname{Exp}_{q_e}\circ T_{[\tau v_{q_e}]}\sigma)(T_{[\tau v_{q_e}]}U)\oplus(T_{\sigma([\tau v_{q_e}])}\operatorname{Exp}_{q_e})(\mathfrak{k}_0\cdot\sigma([\tau v_{q_e}])) \\
&= T_{[\tau v_{q_e}]}\overline{\sigma}(T_{[\tau v_{q_e}]}U)\oplus\mathfrak{k}_0\cdot\overline{\sigma}([\tau v_{q_e}]).
\end{aligned}
$$

Introducing (5.18) in (5.17) and taking into account that $\mathfrak{t}=\mathfrak{k}_0\oplus\mathfrak{k}_1$ we get the statement of the proposition. □

We want to find pairs (v_{q_e},μ) such that $\mathbf{d}V_{\beta(\tau,\mu)}(\operatorname{Exp}_{q_e}(\tau v_{q_e}))=0$ for $\tau>0$. Since $V_{\beta(\tau,\mu)}$ is $G_{\beta(\tau,\mu)}$-invariant, this condition will hold if we only verify it on a subspace of $T_{\operatorname{Exp}_{q_e}(\tau v_{q_e})}Q$ complementary to $\mathfrak{g}_{\beta(\tau,\mu)}\cdot\operatorname{Exp}_{q_e}(\tau v_{q_e})=\mathfrak{t}\cdot\operatorname{Exp}_{q_e}(\tau v_{q_e})$. The previous decomposition of the tangent space immediately yields the following result.

Corollary 5.14. *Suppose that $\mu\in\mathfrak{g}^*$ is such that $\mathfrak{g}_{\beta(\tau,\mu)}=\mathfrak{t}$ for all τ in a neighborhood of zero. Let U and σ be as in Proposition 5.13, $[v_{q_e}]\in U$, and $\overline{\sigma}:=\operatorname{Exp}_{q_e}\circ\sigma$. Then there is an $\epsilon>0$ such that $\mathbf{d}V_{\beta(\tau,\mu)}(\overline{\sigma}([\tau v_{q_e}])=0$ if and only if $\mathbf{d}(V_{\beta(\tau,\mu)}\circ\overline{\sigma})([\tau v_{q_e}])=0$ and $\mathbf{d}(V_{\beta(\tau,\mu)}\circ\operatorname{Exp}_{q_e})(\sigma([\tau v_{q_e}]))|_{\mathfrak{k}_2\cdot q_e}=0$ for $0<\tau<\epsilon$.*

5.7. **The study of two auxiliary functions.** Let I be an open interval containing zero. Recall that $p=\dim\mathfrak{g}_{q_e}=\dim\mathfrak{m}_0$. Let ϑ_1 be an element of a basis $\{\vartheta_1,\vartheta_2,...,\vartheta_p\}$ for $\mathfrak{m}_0$ and define $\beta:(I\setminus\{0\})\times(\mathfrak{m}_1\oplus\mathfrak{m}_2)\to\mathfrak{g}^*$ by

$$\beta(\tau,\mu)=\Pi_1\mu+\tau\Pi_2\mu+\tau^2\vartheta_1,$$

where $\Pi_1:\mathfrak{g}^*\to\mathfrak{m}_1=\mathbb{I}(q_e)\mathfrak{t}$ and $\Pi_2:\mathfrak{g}^*\to\mathfrak{m}_2=\mathfrak{t}^{\circ}$. Notice that this function is a particular case of

$$\beta(\tau,\mu)=\Pi_1\mu+\tau\beta'(\mu)+\tau^2\beta''(\mu),$$

by choosing $\beta'(\mu)=\Pi_2\mu$ and $\beta''(\mu)=\vartheta_1$. Recall that $\mathbb{I}(q_e)=\mathfrak{m}_1\oplus\mathfrak{m}_2$ by Lemma 5.5 and that $\mathbf{J}_L(\mathfrak{g}\cdot q_e)=\mathbb{I}(q_e)\mathfrak{g}$ from the definition of $\mathbf{J}_L$.

Theorem 5.15. *The smooth function $F_1:(I\setminus\{0\})\times U\times\mathbf{J}_L(\mathfrak{g}\cdot q_e)\to\mathbb{R}$ defined by*

$$F_1(\tau,[v_{q_e}],\mu):=(V_{\beta(\tau,\mu)}\circ\overline{\sigma})(\tau[v_{q_e}]).$$

can be extended to a smooth function on $I\times U\times\mathbf{J}_L(\mathfrak{g}\cdot q_e)$, also denoted by F_1. In addition

$$F_1(\tau,[v_{q_e}],\mu)=F_0(\mu)+\tau^2F(\tau,[v_{q_e}],\mu).$$

where F_0, F are defined on $\mathbf{J}_L(\mathfrak{g}\cdot q_e)$ and on $I\times U\times\mathbf{J}_L(\mathfrak{g}\cdot q_e)$ respectively.

Proof. Denote $v_{q_e}:=\sigma([v_{q_e}])\in B\cap(T_{q_e}Q)_{\{e\}}\setminus\mathcal{Z}_\mu$. One can easily see that

$$(V_{\beta(\tau,\mu)}\circ\overline{\sigma})(\tau[v_{q_e}])=V(\operatorname{Exp}_{q_e}(\tau v_{q_e}))+\frac{1}{2}\left\langle\beta(\tau,\mu),\mathbb{I}(\operatorname{Exp}_{q_e}(\tau v_{q_e}))^{-1}\beta(\tau,\mu)\right\rangle.$$

By Remark 5.11, the second term is smooth even in a a neighborhood of $\tau=0$. Since the first term is obviously smooth, it follows that $V_{\beta(\tau,\mu)}\circ\overline{\sigma}$ is smooth also in a neighborhood of $\tau=0$. This is the smooth extension of F_1 in the statement. Let $\{\xi_1,...,\xi_p\}$ be a basis for $\mathfrak{g}_{q_e}\subset\mathfrak{t}$. Then, again by Remark 5.11, we have

$$\mathbb{I}(\operatorname{Exp}_{q_e}(\tau v_{q_e}))^{-1}\beta(\tau,\mu)=\sum_{a=1}^{p}\alpha_a(\tau,v_{q_e},\mu)\xi_a+\eta\left(\tau,v_{q_e},\mu,\sum_{a=1}^{p}\alpha_a(\tau,v_{q_e},\mu)\xi_a\right)$$

where $\alpha_1, ..., \alpha_p, \eta$ are smooth real functions of all their arguments. In what follows we will denote

$$\eta\left(\tau, v_{q_e}, \mu, \sum_{a=1}^{p}\alpha_a(\tau, v_{q_e}, \mu)\xi_a\right) = \eta(\tau, v_{q_e}, \mu, \alpha_1(\tau, v_{q_e}, \mu), ..., \alpha_p(\tau, v_{q_e}, \mu)).$$

Let $\mu \in \mathbf{J}_L(\mathfrak{g}\cdot q_e) = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and $v_{q_e} \in B \cap (T_{q_e}Q)_{\{e\}} \backslash \mathcal{Z}_\mu$. Since in the computations that follow, the arguments v_{q_e} and μ play the role of parameters, we shall denote temporarily $\alpha_a(\tau) = \alpha_a(\tau, v_{q_e}, \mu)$, $a \in \{1, ..., p\}$, and $\eta(\tau, \alpha_1, ..., \alpha_p) = \eta(\tau, v_{q_e}, \mu, \alpha_1(\tau, v_{q_e}, \mu), ..., \alpha_p(\tau, v_{q_e}, \mu))$. Then by (5.11) we get

$$\begin{aligned}\frac{\partial \eta}{\partial \tau}(0, \alpha_1, ..., \alpha_p) = &-\sum_{a=1}^{p}\alpha_a\left(\widehat{\mathbb{I}}(q_e)^{-1}\circ T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\right)\xi_a\\ &-\left(\widehat{\mathbb{I}}(q_e)^{-1}\circ T_{q_e}\widehat{\mathbb{I}}(v_{q_e})\circ\widehat{\mathbb{I}}(q_e)^{-1}\right)\Pi_1\mu + \widehat{\mathbb{I}}(q_e)^{-1}\Pi_2\mu.\end{aligned}$$

Formula (5.3) shows that

$$\frac{\partial \eta}{\partial \alpha_a}(0, \alpha_1, \ldots, \alpha_p) = 0$$

Note that

$$V_{\beta(\tau,\mu)}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))\big|_{\tau=0} = V(q_e) + \frac{1}{2}\left\langle \Pi_1\mu, \widehat{\mathbb{I}}(q_e)^{-1}\Pi_1\mu\right\rangle$$

is independent of v_{q_e}. This shows that $F_1(0, [v_{q_e}], \mu) = F_0(\mu)$ for some smooth function on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. Using Remark 5.11, we get

$$\begin{aligned}\frac{d}{d\tau}\bigg|_{\tau=0} V_{\beta(\tau,\mu)}(\mathrm{Exp}_{q_e}(\tau v_{q_e})) = \mathbf{d}V(q_e)(v_{q_e}) + \frac{1}{2}\left\langle \Pi_2\mu, \sum_{a=1}^{p}\alpha_a(0)\xi_a + \eta(0, \alpha_1, ..., \alpha_p)\right\rangle\\ + \frac{1}{2}\left\langle \Pi_1\mu, \sum_{a=1}^{p}\frac{\partial \alpha_a}{\partial \tau}(0)\left(\xi_a + \frac{\partial \eta}{\partial \alpha_a}(0, \alpha_1, \ldots, \alpha_p)\right) + \frac{\partial \eta}{\partial \tau}(0, \alpha_1, \ldots, \alpha_p)\right\rangle.\end{aligned}$$

The first term $\mathbf{d}V(q_e) = 0$ by Proposition 5.1 (i). Since $\eta(0, v_{q_e}, \mu, \xi) = \eta_\mu = \widehat{\mathbb{I}}(q_e)^{-1}\Pi_1\mu \in \mathfrak{t}$ by Proposition 5.6, we get

$$\sum_{a=1}^{p}\alpha_a(0)\xi_a + \eta(0, \alpha_1, ..., \alpha_p) = \sum_{a=1}^{p}\alpha_a(0)\xi_a + \widehat{\mathbb{I}}(q_e)^{-1}\Pi_1\mu \in \mathfrak{t}.$$

Thus the second term vanishes because $\mathfrak{m}_2 = \mathfrak{t}^\circ$. As $\frac{\partial \eta}{\partial \alpha_a}(0, \alpha_1, \ldots, \alpha_p) = 0$ and $\mathfrak{m}_1$ annihilates $\mathfrak{g}_{q_e}$, the third term becomes

$$\begin{aligned}\left\langle \Pi_1\mu, \frac{\partial \eta}{\partial \tau}(0, \alpha_1, ..., \alpha_p)\right\rangle = &-\sum_{a=1}^{p}\alpha_a\left\langle \Pi_1\mu, \left(\widehat{\mathbb{I}}(q_e)^{-1}\circ T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\right)\xi_a\right\rangle\\ &-\left\langle \Pi_1\mu, \left(\widehat{\mathbb{I}}(q_e)^{-1}\circ T_{q_e}\widehat{\mathbb{I}}(v_{q_e})\circ\widehat{\mathbb{I}}(q_e)^{-1}\right)\Pi_1\mu\right\rangle\\ &+\left\langle \Pi_1\mu, \widehat{\mathbb{I}}(q_e)^{-1}\Pi_2\mu\right\rangle.\end{aligned}$$

We will prove that each summand in this expression vanishes. • Since $\langle \mathfrak{m}_0, \mathfrak{k}_1\rangle = 0$, we get

$$\begin{aligned}\left\langle \Pi_1\mu, \left(\widehat{\mathbb{I}}(q_e)^{-1}\circ T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\right)\xi_a\right\rangle &= \left\langle T_{q_e}\widetilde{\mathbb{I}}(v_{q_e})\xi_a,\ \widehat{\mathbb{I}}(q_e)^{-1}\Pi_1\mu\right\rangle\\ &= \left\langle T_{q_e}\mathbb{I}(v_{q_e})\xi_a,\ \widehat{\mathbb{I}}(q_e)^{-1}\Pi_1\mu\right\rangle = \mathbf{d}\left\langle \mathbb{I}(\cdot)\xi_a, \eta_\mu\right\rangle(q_e)(v_{q_e}) = 0\end{aligned}$$

by (5.7) because $\xi_a \in \mathfrak{g}_{q_e}$ and $\eta_\mu \in \mathfrak{t}$. Thus the first summand vanishes. • The second summand equals

$$\left\langle \Pi_1\mu, \left(\widehat{\mathbb{I}}(q_e)^{-1} \circ T_{q_e}\widehat{\mathbb{I}}(v_{q_e}) \circ \widehat{\mathbb{I}}(q_e)^{-1}\right)\Pi_1\mu\right\rangle = \left\langle T_{q_e}\widehat{\mathbb{I}}(v_{q_e})\eta_\mu, \eta_\mu\right\rangle = \langle T_{q_e}\mathbb{I}(v_{q_e})\eta_\mu, \eta_\mu\rangle$$

because $\langle \mathfrak{m}_0, \mathfrak{k}_1\rangle = 0$. We shall prove that this term vanishes in the following way. Recall that $\eta_\mu \in \mathfrak{k}_1 \subset \mathfrak{t}$. For any $\zeta \in \mathfrak{t}$, hypothesis **(H)** states that $\zeta_Q(q_e)$ is a relative equilibrium and thus, by the augmented potential criterion (see Proposition 3.3), $\mathbf{d}V_\zeta(q_e) = 0$. Since

$$\mathbf{d}V_\zeta(q_e)(u_{q_e}) = \mathbf{d}V(q_e)(u_{q_e}) - \frac{1}{2}\langle T_{q_e}\mathbb{I}(u_{q_e})\zeta, \zeta\rangle$$

for any $u_{q_e} \in T_{q_e}Q$ and $\mathbf{d}V(q_e) = 0$ by Proposition 5.1 (i), it follows that $\langle T_{q_e}\mathbb{I}(u_{q_e})\zeta, \zeta\rangle = 0$. Thus the second summand vanishes. • The third summand is

$$\left\langle \Pi_1\mu, \widehat{\mathbb{I}}(q_e)^{-1}\Pi_2\mu\right\rangle = \langle \Pi_2\mu, \eta_\mu\rangle = 0$$

because $\mathfrak{m}_2 = \mathfrak{t}^\circ$. So, we finally conclude that

$$\left.\frac{d}{d\tau}\right|_{\tau=0} V_{\beta(\tau,\mu)}(\mathrm{Exp}_{q_e}(\tau v_{q_e})) = 0$$

and hence, by Taylor's theorem, we have

$$F_1(\tau, [v_{q_e}], \mu) = F_0(\mu) + \tau^2 F(\tau, [v_{q_e}], \mu)$$

for some smooth function F. □

Theorem 5.16. *The smooth function $G_1 : (I \setminus \{0\}) \times U \times \mathbf{J}_L(\mathfrak{g}\cdot q_e) \to \mathfrak{k}_2^*$ defined by*

$$\langle G_1(\tau, [v_{q_e}], \mu), \varsigma\rangle = \mathbf{d}(V_{\beta(\tau,\mu)} \circ \mathrm{Exp}_{q_e})(\sigma(\tau[v_{q_e}]))\big(\varsigma_Q(q_e)\big), \quad \varsigma \in \mathfrak{k}_2,$$

can be smoothly extended to a function on $I \times U \times \mathbf{J}_L(\mathfrak{g}\cdot q_e)$, also denoted by G_1. In addition,

$$G_1(\tau, [v_{q_e}], \mu) = \tau G(\tau, [v_{q_e}], \mu)$$

where $G : I \times U \times \mathbf{J}_L(\mathfrak{g}\cdot q_e) \to \mathfrak{k}_2^*$ is a smooth function.

Proof. We will show that G_1 is a smooth function at $\tau = 0$ and that $G_1(0, [v_{q_e}], \mu) = 0$. Let $v_{q_e} = \sigma([v_{q_e}])$. Then

$$\begin{aligned}
\langle G_1(\tau, [v_{q_e}], \mu), \varsigma\rangle &= \mathbf{d}V_{\beta(\tau,\mu)}\left(\mathrm{Exp}_{q_e}(\tau v_{q_e})\right)\left(T_{\tau v_{q_e}}\mathrm{Exp}_{q_e}\left(\varsigma_Q(q_e)\right)\right) \\
&= \mathbf{d}V(\mathrm{Exp}_{q_e}(\tau v_{q_e}))\left(T_{\tau v_{q_e}}\mathrm{Exp}_{q_e}\left(\varsigma_Q(q_e)\right)\right) \\
&\qquad + \frac{1}{2}\left\langle \beta(\tau,\mu),\ T_{\mathrm{Exp}_{q_e}(\tau v_{q_e})}(\mathbb{I}(\cdot)^{-1})\left(T_{\tau v_{q_e}}\mathrm{Exp}_{q_e}\left(\varsigma_Q(q_e)\right)\right)\beta(\tau,\mu)\right\rangle \\
&= \mathbf{d}V\left(\mathrm{Exp}_{q_e}(\tau v_{q_e})\right)\left(T_{\tau v_{q_e}}\mathrm{Exp}_{q_e}\left(\varsigma_Q(q_e)\right)\right) - \frac{1}{2}\Big\langle \beta(\tau,\mu), \\
&\qquad \left[\mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))^{-1} \circ T_{\mathrm{Exp}_{q_e}(\tau v_{q_e})}\mathbb{I}\left(T_{\tau v_{q_e}}\mathrm{Exp}_{q_e}\left(\varsigma_Q(q_e)\right)\right) \circ \mathbb{I}(\mathrm{Exp}_{q_e}(\tau v_{q_e}))^{-1}\right]\beta(\tau,\mu)\Big\rangle \\
&= \mathbf{d}V\left(\mathrm{Exp}_{q_e}(\tau v_{q_e})\right)\left(T_{\tau v_{q_e}}\mathrm{Exp}_{q_e}\left(\varsigma_Q(q_e)\right)\right) \\
&\qquad - \frac{1}{2}\left\langle \zeta(\tau, v_{q_e}, \mu),\ T_{\mathrm{Exp}_{q_e}(\tau v_{q_e})}\mathbb{I}\left(T_{\tau v_{q_e}}\mathrm{Exp}_{q_e}\left(\varsigma_Q(q_e)\right)\right)\zeta(\tau, v_{q_e}, \mu)\right\rangle,
\end{aligned}$$

where $\zeta(\tau, v_{q_e}, \mu) := \mathbb{I}^{-1}((\mathrm{Exp}_{q_e}(\tau v_{q_e}))\beta(\tau,\mu)$. Since $\zeta(\tau, v_{q_e}, \mu)$ is smooth in all variables also at $\tau = 0$ by Remark 5.11, it follows that $\langle G_1(\tau, [v_{q_e}], \mu), \varsigma\rangle$ is a smooth function of all its variables. This expression

at $\tau = 0$ equals

$$\begin{aligned}\langle G_1(0,[v_{q_e}],\mu),\varsigma\rangle &= \mathbf{d}V(q_e)(\varsigma_Q(q_e)) - \frac{1}{2}\left\langle \zeta(0,v_{q_e},\mu),\ T_{q_e}\mathbb{I}\left(\varsigma_Q(q_e)\right)\zeta(0,v_{q_e},\mu)\right\rangle \\ &= \mathbf{d}V(q_e)(\varsigma_Q(q_e)) - \frac{1}{2}\left\langle (\mathbb{I}(q_e)[\zeta(0,v_{q_e},\mu),\varsigma],\zeta(0,v_{q_e},\mu)\right\rangle - \frac{1}{2}\left\langle \mathbb{I}(q_e)\zeta(0,v_{q_e},\mu),[\zeta(0,v_{q_e},\mu),\varsigma]\right\rangle \\ &= \mathbf{d}V(q_e)(\varsigma_Q(q_e)) - \langle \mathbb{I}(q_e)\zeta(0,v_{q_e},\mu),[\zeta(0,v_{q_e},\mu),\varsigma]\rangle\end{aligned}$$

by (2.3). Since V is G-invariant it follows that $\mathbf{d}V(q_e)(\varsigma_Q(q_e)) = 0$. Since $\zeta(0,v_{q_e},\mu) = \xi(0,v_{q_e},\mu)+\eta_\mu \in \mathfrak{g}_{q_e}\oplus\mathfrak{k}_1 = \mathfrak{t}$ (see Remark 5.11) it follows that $[\zeta(0,v_{q_e},\mu),\varsigma]\in[\mathfrak{t},\mathfrak{g}]$. By Proposition 5.1 (ii), we have $\mathbb{I}(q_e)\mathfrak{t}\subset[\mathfrak{g},\mathfrak{t}]^\circ$ and hence the second term above also vanishes. Thus we get $\langle G_1(0,[v_{q_e}],\mu),\varsigma\rangle = 0$ for any $\varsigma\in\mathfrak{k}_2$, that is, $G_1(0,[v_{q_e}],\mu) = 0$ which proves the theorem. □

5.8. **Bifurcating branches of relative equilibria.** Let $(Q,\langle\!\langle\cdot,\cdot\rangle\!\rangle_Q,V,G)$ be a simple mechanical G-system, with G a compact Lie group with the Lie algebra $\mathfrak{g}$. Let $q_e\in Q$ be a symmetric point whose isotropy group G_{q_e} is contained in a maximal torus $\mathbb{T}$ of G. Denote by $\mathfrak{t}\subset\mathfrak{g}$ the Lie algebra of $\mathbb{T}$. Let $B\subset(\mathfrak{g}\cdot q_e)^\perp$ be a G_{q_e}–invariant open neighborhood of $0_{q_e}\in(\mathfrak{g}\cdot q_e)^\perp$ such that the exponential map is injective on B and for any $q\in G\cdot\operatorname{Exp}_{q_e}(B)$ the isotropy subgroup G_q is conjugate to a (not necessarily proper) subgroup of G_{q_e}. Define the closed G_{q_e}–invariant subset $\mathcal{Z}_{\mu^0} =: \{v_{q_e}\in B\cap(T_{q_e}Q)_{\{e\}} \mid \det A = 0\}$, where $\mu^0\in\mathfrak{m}_1\oplus\mathfrak{m}_2$ is arbitrarily chosen and the entries of the matrix A are given in (5.12). Let $U\subset[B\cap(T_{q_e}Q)_{\{e\}}\setminus\mathcal{Z}_{\mu^0}]/G_{q_e}$ be open and consider the functions F and G given in Theorems 5.15 and 5.16. Define $G^i : I\times U\times(\mathfrak{m}_1\oplus\mathfrak{m}_2)\to\mathbb{R}$ by

$$G^i(\tau,[v_{q_e}],\mu_1+\mu_2) := \langle G(\tau,[v_{q_e}],\mu_1+\mu_2),\varsigma_i\rangle,$$

where $\{\varsigma_i \mid i = 1,...,\dim\mathfrak{k}_2\}$ is a basis for $\mathfrak{k}_2$. Choose $([v_{q_e}],\mu_1+\mu_2)\in U\times(\mathfrak{m}_1\oplus\mathfrak{m}_2)$ such that

$$\frac{\partial F}{\partial u}(0,[v_{q_e}],\mu_1+\mu_2) = 0,$$

where the partial derivative is taken relative to the variable $u\in U$. Define the matrix

$$\Delta_{([v_{q_e}],\mu_1,\mu_2)} := \begin{bmatrix} \frac{\partial^2 F}{\partial u^2}(0,[v_{q_e}],\mu_1+\mu_2) & \frac{\partial^2 F}{\partial\mu_2\partial u}(0,[v_{q_e}],\mu_1+\mu_2) \\ \frac{\partial G^i}{\partial u}(0,[v_{q_e}],\mu_1+\mu_2) & \frac{\partial G^i}{\partial\mu_2}(0,[v_{q_e}],\mu_1+\mu_2)\end{bmatrix},$$

where the partial derivatives are evaluated at $\tau = 0, [v_{q_e}], \mu = \mu_1+\mu_2$. Here $\frac{\partial}{\partial\mu_2}$ denotes the partial derivative with respect to the $\mathfrak{m}_2$-component μ_2 of μ. In the framework and the notations introduced above we will state and prove the main result of this paper. Let $\pi : TQ\to(TQ)/G$ be the canonical projection and $\mathcal{R}_e := \pi(\mathfrak{t}\cdot q_e)$.

Theorem 5.17. *Assume the following:*

(**H**) *every $v_{q_e}\in\mathfrak{t}\cdot q_e$ is a relative equilibrium.*

If there is a point $([v^0_{q_e}],\mu^0_1+\mu^0_2)\in U\times(\mathfrak{m}_1\oplus\mathfrak{m}_2)$ such that

1) $\frac{\partial F}{\partial u}(0,[v^0_{q_e}],\mu^0_1+\mu^0_2) = 0,$
2) $G^i(0,[v^0_{q_e}],\mu^0_1+\mu^0_2) = 0$
3) $\Delta_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}$ *is nondegenerate,*

then there exists a family of continuous curves $\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)} : [0,1]\to(TQ)/G$ parameterized by μ_1 in a small neighborhood $\mathcal{V}_0$ of μ^0_1 consisting of classes of relative equilibria with trivial isotropy on

$\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(0,1)$ *satisfying*

$$\operatorname{Im}\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}\bigcap\mathcal{R}_e=\left\{\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(0)\right\}$$

and $\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(0)=[\zeta_Q(q_e)]$*, where* $\zeta=\widehat{\mathbb{I}}(q_e)^{-1}\mu_1\in\mathfrak{t}$*. For* $\mu_1,\mu_1'\in\mathcal{V}_0$ *with* $\mu_1\neq\mu_1'$*, where* $\mathcal{V}_0$ *is as above, the above branches do not intersect, that is,*

$$\left\{\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(\tau)\;\Big|\;\tau\in[0,1]\right\}\bigcap\left\{\gamma^{\mu_1'}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(\tau)\;\Big|\;\tau\in[0,1]\right\}=\varnothing.$$

Suppose that $([v^0_{q_e}],\mu^0_1,\mu^0_2)\neq([v^1_{q_e}],\mu^1_1,\mu^1_2)$*. (i) If* $\mu^0_1\neq\mu^1_1$ *then the families of relative equilibria do not intersect, that is,*

$$\left\{\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(\tau)\;\Big|\;(\tau,\mu_1)\in[0,1]\times\mathcal{V}_0\right\}\bigcap\left\{\gamma^{\mu_1'}_{([v^1_{q_e}],\mu^1_1,\mu^1_2)}(\tau)\;\Big|\;(\tau,\mu_1')\in[0,1]\times\mathcal{V}_1\right\}=\varnothing,$$

where $\mathcal{V}_0$ *and* $\mathcal{V}_1$ *are two small neighborhoods of* μ^0_1 *and* μ^1_1 *respectively such that* $\mathcal{V}_0\cap\mathcal{V}_1=\varnothing$*. (ii) If* $\mu^0_1=\mu^1_1=\overline{\mu}$ *and* $[v^0_{q_e}]\neq[v^1_{q_e}]$ *then* $\gamma^{\overline{\mu}}_{([v^0_{q_e}],\overline{\mu},\mu^0_2)}(0)=\gamma^{\overline{\mu}}_{([v^1_{q_e}],\overline{\mu},\mu^1_2)}(0)$ *and for* $\tau>0$ *we have*

$$\left\{\gamma^{\overline{\mu}}_{([v^0_{q_e}],\overline{\mu},\mu^0_2)}(\tau)\;\Big|\;\tau\in(0,1]\right\}\bigcap\left\{\gamma^{\overline{\mu}}_{([v^1_{q_e}],\overline{\mu},\mu^1_2)}(\tau)\;\Big|\;\tau\in(0,1]\right\}=\varnothing.$$

Proof. Let $([v^0_{q_e}],\mu^0_1+\mu^0_2)\in U\times(\mathfrak{m}_1\oplus\mathfrak{m}_2)$ be such that the conditions 1-3 hold. Because $\Delta_{([v^0_{q_e}],\mu^0_1+\mu^0_2)}$ is nondegenerate, we can apply the implicit function theorem for the system $(\frac{\partial F}{\partial u},G^i)(\tau,[v_{q_e}],\mu_1+\mu_2)=0$ around the point $(0,[v^0_{q_e}],\mu^0_1+\mu^0_2)$ and so we can find an open neighborhood $J\times\mathcal{V}_0$ of the point $(0,\mu^0_1)$ in $I\times\mathfrak{m}_1$ and two functions $u:J\times\mathcal{V}_0\to U$ and $\mu_2:J\times\mathcal{V}_0\to\mathfrak{m}_2$ such that $u(0,\mu^0_1)=[v^0_{q_e}]$, $\mu_2(0,\mu^0_1)=\mu^0_2$ and

$$\begin{array}{ll} i) & \frac{\partial F}{\partial u}(\tau,u(\tau,\mu_1),\mu_1+\mu_2(\tau,\mu_1))=0\\ ii) & G^i(\tau,u(\tau,\mu_1),\mu_1+\mu_2(\tau,\mu_1))=0.\end{array}$$

Therefore, from Theorems 5.15 and 5.16 it follows that the relative equilibrium conditions of Corollary 5.14 are both satisfied. Thus we obtain the following family of branches of relative equilibria $[(\overline{\sigma}(\tau\cdot u(\tau,\mu_1)),\beta(\tau,\mu_1+\mu_2(\tau,\mu_1)))]_G$ parameterized by $\mu_1\in\mathcal{V}_0$. For $\tau>0$ the isotropy subgroup is trivial and for $\tau=0$ the corresponding points on the branches are $[(\overline{\sigma}([0_{q_e}]),\mu_1]_G=[q_e,\mu_1]_G$ which have the isotropy subgroup equal to G_{q_e}. This shows that there are points in $\mathcal{R}_e$ from which there are emerging branches of relative equilibria with broken trivial symmetry. Using now the correspondence given by Proposition 5.3 and a rescaling of τ we obtain the desired family of continuous curves $\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}:[0,1]\to(TQ)/G$ parameterized by μ_1 in a small neighborhood $\mathcal{V}_0$ of μ^0_1 consisting of classes of relative equilibria with trivial isotropy on $\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(0,1)$ and such that

$$\operatorname{Im}\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}\bigcap\mathcal{R}_e=\{\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(0)\}$$

and $\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(0)=[\zeta_Q(q_e)]$, where $\zeta=\widehat{\mathbb{I}}(q_e)^{-1}\mu_1$. Equivalently, using the identification given by (5.16) and by Proposition 5.3 we obtain that the branches of relative equilibria $\gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(\tau)\in(TQ)/G$ are identified with $[\sigma(\tau\cdot u(\tau,\mu_1)),\beta(\tau,\mu_1+\mu_2(\tau,\mu_1))]_{G_{q_e}}$. It is easy to see that for $\mu_1\neq\mu_1'$ we have that

$\beta(\tau, \mu_1 + \mu_2(\tau, \mu_1)) \neq \beta(\tau', \mu_1' + \mu_2(\tau, \mu_1'))$ for every $\tau, \tau' \in [0,1]$. Using now the fact that G_{q_e} acts trivially on $\mathfrak{m}_1$ we obtain

$$\left\{ \gamma^{\mu_1}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(\tau) \,\middle|\, \tau \in [0,1] \right\} \bigcap \left\{ \gamma^{\mu_1'}_{([v^0_{q_e}],\mu^0_1,\mu^0_2)}(\tau) \,\middle|\, \tau \in [0,1] \right\} = \varnothing.$$

In an analogous way, using the same argument we can prove (i). For (ii) we start with two branches of relative equilibria, $b_1(\tau, \overline{\mu}) := [\sigma(\tau \cdot u(\tau, \overline{\mu})), \beta(\tau, \overline{\mu} + \mu_2(\tau, \overline{\mu}))]_{G_{q_e}}$ and $b_2(\tau', \overline{\mu}) := [\sigma(\tau' \cdot u'(\tau', \overline{\mu})), \beta(\tau', \overline{\mu} + \mu_2(\tau, \overline{\mu}))]_{G_{q_e}}$. For $\tau = \tau' = 0$ we have $b_1(0, \overline{\mu}) = [0, \overline{\mu}]_{G_{q_e}} = b_2(0, \overline{\mu})$. We also have $u(0, \overline{\mu}) = [v^0_{q_e}] \neq [v^1_{q_e}] = u'(0, \overline{\mu})$ and so, from the implicit function theorem, we obtain $u(\tau, \overline{\mu}) \neq u'(\tau', \overline{\mu})$ for $\tau, \tau' > 0$ small enough. Suppose that there exist $\tau, \tau' > 0$ such that $b_1(\tau, \overline{\mu}) = b_2(\tau', \overline{\mu})$. Then using the triviality of the G_{q_e}-action on $\mathfrak{m}_0$ we obtain that $\tau^2 \nu_0 = {\tau'}^2 \nu_0$ and consequently $\tau = \tau'$. The conclusion of (ii) follows now by rescaling. $\square$

Remark 5.18. We can have two particular forms for the rescaling β according to special choices of the groups G and G_{q_e}, respectively. (a) If G is a torus, then from the splitting $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2$, where $\mathfrak{k}_0 = \mathfrak{g}_{q_e}$, $\mathfrak{k}_0 \oplus \mathfrak{k}_1 = \mathfrak{t}$, and $\mathfrak{k}_2 = [\mathfrak{g}, \mathfrak{t}]$, we conclude that $\mathfrak{k}_2 = \{0\}$ (since $\mathfrak{g} = \mathfrak{t}$) and consequently $\mathfrak{m}_2 = \{0\}$. In this case we will obtain the special form for the rescaling $\beta : I \times \mathfrak{m}_1 \to \mathfrak{g}^*$, $\beta(\tau, \mu) = \mu + \tau^2 \nu_0$. (b) If is G_{q_e} a maximal torus in G, so $\mathfrak{g}_{q_e} = \mathfrak{t}$, then the same splitting implies that $\mathfrak{k}_1 = \{0\}$ and consequently $\mathfrak{m}_1 = \{0\}$. In this case we will obtain the special form for the rescaling $\beta : I \times \mathfrak{m}_2 \to \mathfrak{g}^*$, $\beta(\tau, \mu) = \tau\mu + \tau^2 \nu_0$.

6. Stability of the bifurcating branches of relative equilibria

In this section we shall study the stability of the branches of relative equilibria found in the previous section. We will do this by applying a result of Patrick [16] on G_μ-stability to our situation. First we shortly review this result.

Definition 6.1. *Let z_e be a relative equilibrium with velocity ξ_e and $J(z_e) = \mu_e$. We say that z_e is **formally stable** if $\mathbf{d}^2(H - J^{\xi_e})(z_e)|_{T_{z_e} J^{-1}(\mu_e)}$ is a positive or negative definite quadratic form on some (and hence any) complement to $\mathfrak{g}_{\mu_e} \cdot z_e$ in $T_{z_e} J^{-1}(\mu_e)$.*

We have the following criteria for formal stability.

Theorem 6.2 (Patrick, 1995)**.** *Let $z_e \in T^*Q$ be a relative equilibrium with momentum value $\mu_e \in \mathfrak{g}^*$ and base point $q_e \in Q$. Assume that $\mathfrak{g}_{q_e} = \{0\}$. Then z_e is formally stable if and only if $\mathbf{d}^2 V_\mu(q_e)$ is positive definite on one (and hence any) complement $\mathfrak{g}_\mu \cdot q_e$ in $T_{q_e}Q$.*

To apply this theorem to our case in order to obtain the formal stability of the relative equilibria on a bifurcating branch we proceed as follows. First notice that if we fix $\mu \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and $[v_{q_e}] \in U$ as in Theorem 5.17, we obtain locally a branch of relative equilibria with trivial isotropy bifurcating from our initial set. More precisely, this branch starts at the point

$$\left(\widehat{\mathbb{I}}(q_e)^{-1} \Pi_1 \mu \right)_Q (q_e).$$

The momentum values along this branch are $\beta(\tau, \mu)$, and for $\tau \neq 0$ the velocities have the expression $\mathbb{I}(\mathrm{Exp}_{q_e}(\sigma(\tau u(\tau, \mu_1))^{-1} \beta(\tau, \mu)$. The base points of this branch are $\mathrm{Exp}_{q_e}(\sigma(\tau u(\tau, \mu_1))$. Recall from Corollary 5.14 that we introduced the notation $\overline{\sigma} := \mathrm{Exp}\, q_e \circ \sigma$ that will be used below. By the definition of $\beta(\tau, \mu)$ we have $\mathfrak{g}_{\beta(\tau,\mu)} = \mathfrak{t}$ for all τ, even for $\tau = 0$. The base points for the entire branch have no symmetry for $\tau > 0$ so we can characterize the formal stability (in our case the $\mathbb{T}$ -stability) of the whole branch (locally) in terms of Theorem 6.2. We begin by giving sufficient conditions that

guarantee the $\mathbb{T}$-stability of the branch, since $G_{\beta(\tau,\mu)} = \mathbb{T}$. To do this, one needs to find conditions that insure that for $\tau \neq 0$ (where the amended potential exists)

$$\mathbf{d}^2 V_{\beta(\tau,\mu)}(\overline{\sigma}(\tau u(\tau,\mu_1)))|_{T_{[\tau u(\tau,\mu_1)]}\overline{\sigma}(T_{[\tau u(\tau,\mu_1)]}U)\oplus(T_{\sigma([\tau u(\tau,\mu_1)])}\operatorname{Exp}_{q_e})(\mathfrak{k}_2\cdot q_e)}$$

is positive definite. We do not know how to control the cross terms of this quadratic form. This is why we shall work only with Abelian groups G since in that case the subspace $\mathfrak{k}_2 = \{0\}$ and the second summand thus vanishes. So, let G be a torus $\mathbb{T}$. By Proposition 5.13 and Theorem 5.15, the second variation

$$\mathbf{d}^2 V_{\beta(\tau,\mu)}(\overline{\sigma}(\tau u(\tau,\mu_1)))|_{T_{[\tau u(\tau,\mu_1)]}\overline{\sigma}(T_{[\tau u(\tau,\mu_1)]}U)}$$

coincides for $\tau \neq 0$, with the second variation

$$\mathbf{d}^2_U F_1(\tau, u(\tau,\mu_1), \mu_1 + \mu_2(\tau,\mu_1))|_{T_{[\tau u(\tau,\mu_1)]}U} \tag{6.1}$$

of the auxiliary function F_1, where $\mathbf{d}^2_U$ denotes the second variation relative to the second variable in F_1. But, unlike $V_{\beta(\tau,\mu)}$, the function F_1 is is defined even at $\tau = 0$. Recall from Theorem 5.15 that on the bifurcating branch the amended potential has the expression

$$F_1(\tau, u(\tau,\mu_1), \mu_1 + \mu_2(\tau,\mu_1)) = F_0(\mu_1 + \mu_2(\tau,\mu_1)) + \tau^2 F(\tau, u(\tau,\mu_1), \mu_1 + \mu_2(\tau,\mu_1)),$$

where F_0 is smooth on $\mathbf{J}_L(\mathfrak{g}\cdot q_e) = \mathbb{I}(q_e)\mathfrak{g}$ and F, F_1 are both smooth functions on $I \times U \times \mathbf{J}_L(\mathfrak{g}\cdot q_e)$, even around $\tau = 0$. So, if the second variation of F at $(0, [v^0_{q_e}], \mu^0_1 + \mu^0_2)$ is positive definite, then the quadratic form (6.1) will remain positive definite along the branch for $\tau > 0$ small. So we get the following result.

Theorem 6.3. *Let $\mu^0_1 + \mu^0_2 \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and $[v^0_{q_e}] \in U$ be as in the Theorem 5.17 and assume that $\mathbf{d}^2_U F(0, [v^0_{q_e}], \mu^0_1 + \mu^0_2)$ is positive definite. Then the branch of relative equilibria with no symmetry which bifurcate form $\left(\widehat{\mathbb{I}}(q_e)^{-1}\mu^0_1\right)_Q(q_e)$ will be $\mathbb{T}$-stable for $\tau > 0$ small.*

A direct application of this criterion to the double spherical pendulum recovers the stability result on the bifurcating branches proved directly in [13].

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